$\frac{\infty}{\infty}$ or $\frac{0}{0}$

Review of Sections 4.4 and 4.7

1. Find the limit.

(a) $\lim_{x\to\infty} \frac{(\ln x)^2}{x-1}$ $\left|\frac{\infty}{\infty}\right|$ $\lim_{x\to\infty} \frac{2(\ln x)\cdot\frac{1}{x}}{1} = 2\lim_{x\to\infty}\frac{\ln x}{x}$ $\left|\frac{\infty}{\infty}\right|$ $= 2\lim_{x\to\infty}\frac{\frac{1}{x}}{1}$

$= 2\lim_{x\to\infty}\frac{1}{x} = 0$

(a)' $\lim_{x\to 1}\frac{(\ln x)^2}{x-1} = \left|\frac{0}{0}\right| = \lim_{x\to 1}\frac{2\ln x\ \frac{1}{x}}{1} = 2\lim_{x\to 1}\frac{\ln x}{x} = 2\ \frac{\ln 1}{1} = 0.$

(b) $\lim_{x\to 0}\frac{\sin x - x}{x^3} = \left|\frac{0}{0}\right| = \lim_{x\to 0}\frac{\cos x - 1}{3x^2}$ $\left|\frac{0}{0}\right| = \lim_{x\to 0}\frac{-\sin x}{6x}$ $\left|\frac{0}{0}\right|$

$= \lim_{x\to 0}\frac{-\cos x}{6} = \frac{-\cos 0}{6} = \boxed{-\frac{1}{6}}$

(c) $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$ $|\infty-\infty| = \lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}$ $\left|\frac{0}{0}\right|$

differentiate

$$= \lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}} = \lim_{x\to 1}\frac{\frac{x-1}{x}}{\frac{x\ln x+(x-1)}{x}}$$

$$= \lim_{x\to 1}\frac{x-1}{\cancel{x}}\cdot\frac{\cancel{x}}{x\ln x+(x-1)} = \lim_{x\to 1}\frac{x-1}{x-1+x\ln x} \quad \left|\frac{0}{0}\right|$$

$$= \lim_{x\to 1}\frac{1}{1+\ln x+x\cdot\frac{1}{x}} = \lim_{x\to 1}\frac{1}{1+\ln x+1} = \lim_{x\to 1}\frac{1}{2+\ln x}$$

$$= \frac{1}{2+\cancel{\ln 1}\,0} = \boxed{\frac{1}{2}}$$

(d) $\lim_{x\to 0^+} x^2 \ln x = |0\cdot\infty| = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x^2}} = \left|\frac{\infty}{\infty}\right|$ differentiate $\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{2}{x^3}}$

$$= \lim_{x\to 0^+}\frac{1}{\cancel{x}}\left(-\frac{x^{\cancel{3}\,2}}{2}\right) = \lim_{x\to 0^+}\left(-\frac{1}{2}x^2\right) = \boxed{0}$$

(d') $\lim_{x\to\infty} x\tan\frac{7}{x}$ $|\infty\cdot 0| = \lim_{x\to\infty}\frac{\tan\frac{7}{x}}{\frac{1}{x}}$ $\left|\frac{0}{0}\right|$

derivative

$$\lim_{x\to\infty}\frac{\sec^2\frac{7}{x}\left(-\frac{7}{\cancel{x^2}}\right)}{-\frac{1}{\cancel{x^2}}} = 7\lim_{x\to\infty}\sec^2\frac{7}{x}$$

$$= 7\lim_{x\to\infty}\frac{1}{\cos^2\frac{7}{x}\to 0} = 7\,\frac{1}{\cancel{\cos^2 0}\to 1} = \boxed{7}$$

(e) $\lim_{x\to\infty} \sqrt{x}e^{-x/2}$ $|0\cdot\infty| = \lim_{x\to\infty} \frac{\sqrt{x}}{e^{x/2}}$ $\left|\frac{\infty}{\infty}\right| \overset{\text{derivative}}{=} \lim_{x\to\infty} \frac{\frac{1}{\cancel{2}\sqrt{x}}}{\frac{1}{\cancel{2}}e^{x/2}}$

$= \lim_{x\to\infty} \frac{1}{\sqrt{x}\,e^{x/2}}$ $\infty$ as $x\to\infty$ $= \boxed{0}$

$$[g(x)]^{f(x)} = e^{f(x)\ln g(x)}$$

(f) $\lim_{x\to 0}(\sin x)^{\tan x}$ $|0^0| = \lim_{x\to 0} e^{\tan x \ln(\sin x)}$

$= e^{\lim_{x\to 0} \tan x \ln(\sin x)}$

$\boxed{\tan x = \frac{1}{\cot x}}$

$\lim_{x\to 0} \tan x \ln(\sin x)$ $|0\cdot\infty| = \lim_{x\to 0} \frac{\ln(\sin x)}{\cot x}$ $\left|\frac{\infty}{\infty}\right|$

differentiate $\lim_{x\to 0} \frac{\frac{1}{\sin x}(\cos x)}{-\csc^2 x} = -\lim_{x\to 0} \frac{\cos x \cdot \sin^{\cancel{2}} x}{\cancel{\sin x}}$

$= -\lim_{x\to 0} \overset{1}{\cancel{\cos x}}\ \overset{0}{\cancel{\sin x}} = 0$

answer $\boxed{e^0 = 1}$

(g) $\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1} = \left|1^{\infty}\right| = \lim_{x\to\infty} e^{(2x+1)\ln\frac{2x-3}{2x+5}}$

$$= e^{\lim_{x\to\infty}(2x+1)\ln\frac{2x-3}{2x+5}}$$

$$\lim_{x\to\infty}(2x+1)\ln\frac{2x-3}{2x+5} \quad |\infty\cdot 0| = \lim_{x\to\infty}(2x+1)\left[\ln(2x-3)-\ln(2x+5)\right]$$

$$= \lim_{x\to\infty}\frac{\ln(2x-3)-\ln(2x+5)}{\frac{1}{2x+1}} \quad \left|\frac{\infty}{\infty}\right| = \lim_{x\to\infty}\frac{\frac{\cancel{2}}{2x-3}-\frac{\cancel{2}}{2x+5}}{-\frac{1}{(2x+1)^2}(\cancel{2})}$$

$$= -\lim_{x\to\infty}(2x+1)^2\,\frac{2x+5-(2x-3)}{(2x-3)(2x+5)} = -\lim_{x\to\infty}\frac{8(2x+1)^2}{(2x-3)(2x+5)} = \left|\frac{\infty}{\infty}\right|$$

$$= -8\lim_{x\to\infty}\frac{2(2x+1)(2)}{2(2x+5)+2(2x-3)} = -8\lim_{x\to\infty}\frac{4(2x+1)}{2\left[2x+5+2x-3\right]} = -8\lim_{x\to\infty}\frac{4(2x+1)}{2(4x+2)}$$

$$= -8\lim_{x\to\infty}\frac{8}{8} = -8$$

$\boxed{\text{answer} \quad e^{-8}}$

(h) $\lim_{x\to 0^+}(1+\sin 3x)^{1/x} \quad \left|1^{\infty}\right| = \lim_{x\to 0^+} e^{\frac{1}{x}\ln(1+\sin 3x)}$

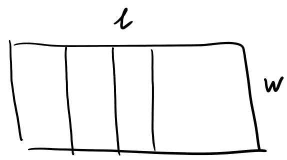
$$= e^{\lim_{x\to 0}\frac{1}{x}\ln(1+\sin 3x)}$$

$$\lim_{x\to 0}\frac{\ln(1+\sin 3x)}{x} \quad \left|\frac{0}{0}\right| = \lim_{x\to 0}\frac{\frac{1}{1+\sin(3x)}(\sin 3x)'}{1}$$

$$= \lim_{x\to 0}\frac{3\cos 3x}{1+\sin(3x)} = \frac{3\cancel{\cos 0}^{\,1}}{1+\cancel{\sin 0}^{\,0}} = 3$$

$\boxed{\text{answer} \quad e^{3}}$

2. A farmer with 750 ft of fencing wants to enclose a rectangular field and then divide it in four parts with a fence parallel to one of the sides of the rectangle. What is the largest possible total area of the four pens?




$l$ is the total length of the field
$w$ is the width of the field.
The area is $\boxed{A = lw}$

$750 = 2l + 5w \Rightarrow 2l = 750 - 5w$

$$\boxed{l = \frac{750-5w}{2}}$$

area $A(w) = \dfrac{750-5w}{2}\,w$

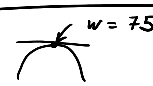
$A(w) = 375w - \frac{5}{2}w^2$

$A'(w) = 375 - 5w = 0 \Rightarrow \boxed{w = \frac{375}{5} = 75}$

$l = \dfrac{750-5(75)}{2} = \dfrac{5(75)}{2} = \dfrac{375}{2}$

$\boxed{w = 75,\ l = \frac{375}{2}}$

2nd derivative test: $A''(w) = -5 < 0$

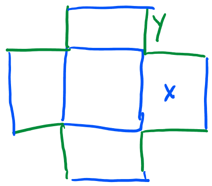



$A(w)$ has a local max @ $w = 75$.

The largest area is

$A(75) = 375(75) - \frac{5}{2}(75)^2 = \boxed{14062.5\ (ft^2)}$

3. A box with a square base and open top must have a volume of 32000 cm$^3$. Find the dimensions of the box that minimize the amount of material used.




y is the height of the box
x is the side of the base.

Minimize the surface area of the box.

$\boxed{S.A. = x^2 + 4xy}$

Volume of the box is $V = \boxed{x^2 \cdot y = 32{,}000}$

$y = \frac{32{,}000}{x^2}$ ← plug into S.A.

$S.A.(x) = x^2 + 4x \underbrace{\frac{32{,}000}{x^2}}$ "y

$S.A.(x) = x^2 + \frac{128{,}000}{x}$ ← minimize

$(S.A.)'(x) = 2x - \frac{128{,}000}{x^2} = 0$ or $(x^2)\frac{2x}{2} = \frac{128{,}000}{2\cancel{x^2}}\cancel{(x^2)}$

$x^3 = 64{,}000$

$\boxed{x = \sqrt[3]{64{,}000} = 40}$, then $\boxed{y = \frac{32{,}000}{x^2} = \frac{32{,}0\cancel{00}}{16\cancel{00}} = 20}$

Check whether S.A.(x) has a local min @ x=40.
2nd derivative test.

$(S.A.)''(x) = 2 - 128{,}000\,(-2)\,x^{-3} = 2 + \frac{256{,}000}{x^3}$

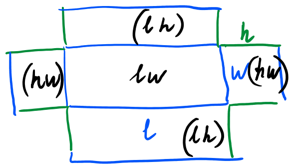
$(S.A.)''(40) = 2 + \frac{256{,}000}{40^3} > 0$

C.U.

x=40.

S.A. has a local min @ x=40

4. A rectangular storage container with an open top is to have a volume of 10 m$^3$. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of the cheapest such container.




The total cost

$$C = (lw)10 + 6(2hw + 2hl)$$

$$= 10lw + 6(2h)(w + l)$$

$$C = 10\overset{2w}{\cancel{l}}w + 12h(w + \overset{2w}{\cancel{l}})$$

$$C = 20w^2 + 12h(3w)$$

$$C = 20w^2 + 36hw$$

The volume $V = \boxed{\overset{2w}{\cancel{l}}wh = 10}$

$l = 2w$

$V = (2w)wh = 10$

$2w^2 h = 10$

$$\boxed{h = \frac{10}{2w^2} = \frac{5}{w^2}}$$

$$C = 20w^2 + \frac{36(5)\cancel{w}}{w^{\cancel{2}}}$$

$$\boxed{C(w) = 20w^2 + \frac{180}{w}} \leftarrow \text{minimize}$$

$$C'(w) = 40w - \frac{180}{w^2} = 0$$

$$w^2 \, 4\cancel{0}w = \frac{18\cancel{0}}{w^2} \, w^2$$

$$w^3 = \frac{18}{4} = \frac{9}{2}$$

$$\boxed{w = \sqrt[3]{\frac{9}{2}}}$$

2nd derivative test: $C''(w) = 40 - 180(-2)w^{-3}$

$$= 40 + \frac{360}{w^3}$$

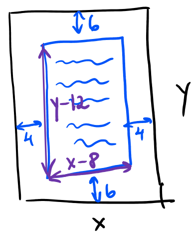
$$C''\left(\sqrt[3]{\frac{9}{2}}\right) = 40 + \frac{360}{\frac{9}{2}} > 0$$

C has a local min @ $w = \sqrt[3]{\frac{9}{2}}$

Total cost $C(w) = 20 \cdot \left(\frac{9}{2}\right)^{2/3} + \frac{180}{\sqrt[3]{\frac{9}{2}}}$

$$\boxed{C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\frac{9}{2}\right)^{2/3} + 180\left(\frac{9}{2}\right)^{-1/3}}$$

5. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm$^2$, find the dimensions of the poster with the smallest area.




$A = xy$

Printed Area $= \boxed{\frac{(y-12)(x-8)}{y-12} = \frac{384}{y-12}}$

solve for x: $x-8 = \frac{384}{y-12}$

$x = 8 + \frac{384}{y-12}$

$$\boxed{x = \frac{8(y-12)+384}{y-12} = \frac{8y+288}{y-12}}$$

Plug x into $A=xy$

$$A(y) = \frac{8y+288}{y-12}\,y = \frac{8y^2+288y}{y-12} = \frac{8(y^2+36y)}{y-12} \leftarrow \text{minimize}$$

$$A'(y) = \frac{8(2y+36)(y-12) - 8(y^2+36y)(1)}{(y-12)^2} = \frac{16(y+18)(y-12) - 8y^2 - 288y}{(y-12)^2} = 0$$

$$\frac{16(y+18)(y-12) - 8(y^2+36y)}{8} = \frac{0}{8}$$

$$2(y+18)(y-12) - y^2 - 36y = 0$$

$$2(y^2 - 12y + 18y - 216) - y^2 - 36y = 0$$

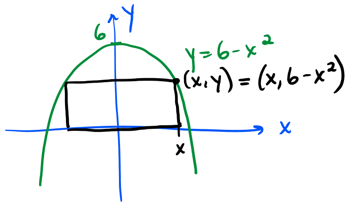
$$2(y^2 + 6y - 216) - y^2 - 36y = 0$$

$$2y^2 + 12y - 432 - y^2 - 36y = 0$$

$$y^2 - 24y - 432 = 0$$

$$y_1 = \frac{24 + \sqrt{24^2 + 4(432)}}{2} = 36$$

6. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 6 - x^2$.




$l = 2x$

$h = y = 6 - x^2$

$A = l\,h = 2x(y) = 2x(6 - x^2)$

$A(x) = 12x - 2x^3$

$A'(x) = 12 - 6x^2 = 0$

$12 = 6x^2$ or $x^2 = 2$ or $x = \sqrt{2}$

$\boxed{l = 2\sqrt{2},\ h = 6 - (\sqrt{2})^2 = 4}$

2nd derivative test : $A''(x) = -12x < 0$ when $x = \sqrt{2}$.

$A(x)$ has a local max @ $\boxed{l = 2\sqrt{2} \text{ and } h = 4}$

$y = 5 - 6x$

7. Find the point on the line $6x + y = 5$ that are closest to the point $(-5, 3)$.

$(x, y)$ on the line

$D^2 = (x - (-5))^2 + (y-3)^2$ distance from $(x,y)$ to $(-5,3)$

$D^2 = (x+5)^2 + (y-3)^2$

$D^2 = (x+5)^2 + (5-6x-3)^2$

$D^2 = (x+5)^2 + (2-6x)^2$

$$\frac{d}{dx}(D^2) = \frac{2(x+5) + 2(2-6x)(-6)}{2} = 0$$

$x + 5 + (2-6x)(-6) = 0$

$x + 5 - 12 + 36x = 0$

$37x - 7 = 0 \Rightarrow \boxed{x = \frac{7}{37}}$

$\boxed{y = 5 - 6 \cdot \frac{7}{37} = \frac{143}{37}}$

2nd derivative test

$\frac{d^2}{dx^2}(D^2) = 2(37) > 0$, thus the distance is a min.

8. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side $L$ if one side of the rectangle lies on the base of a triangle.