Section 1.3

- **Limits at Infinity:** We write $\lim_{x\to\pm\infty} f(x) = L$ if $f(x)$ approaches the number L as x increases (or decreases) without bound. We refer to the line $y = L$ as a **horizontal asymptote**.

- **Limits at Infinity of Polynomial Functions:** To determine the end behavior of a polynomial, we must look at the leading term (i.e. the term with the highest power of x). If, $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$, where n is a postive integer, then $\lim_{x\to\infty} p(x) = \lim_{x\to\infty} a_n x^n = \pm\infty$ and $\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} a_n x^n = \pm\infty$

  Note: We could also remember the information from the table in the Section 1.3 Notes.

- **Limits at Infinity of Rational Functions:** If $f(x)$ is a rational function, to determine the limits at infinity, we divide every term in the function by the highest power of x in the denominator, simplify each term, and then observe the behavior of each term.

  Note: Keep in mind that if the limit at ∞ or at $-\infty$ is finite for a rational function, the limit at the other end is the same finite number. This means that if the graph of a rational function has a horizontal asymptote at one end, the graph will have the same horizontal asymptote at the other end. Please keep in mind that this is true for rational functions but not true in general.

- **Limits at Infinity of Functions Involving Exponentials:** To evaluate limits at infinity of functions involving exponentials you must first be familiar with the graphs of e^x and e^{-x}. This process is slightly different depending on whether you are looking at the limit as $x \to \infty$ or $x \to -\infty$:
  - If $x \to \infty$, divide every term in the function by the most positive power of e^{nx} in the denominator.
  - If $x \to -\infty$, divide every term in the function by the most negative power of e^{nx} in the denominator.
  - Simplify the function and then observe the behavior of each term of the function.
  - Note: If there is no positive exponent of e^{nx} in the denominator when $x \to \infty$ or no negative exponent of e^{nx} in the denominator when $x \to -\infty$, we do not divide by anything. We instead just observe the behavior of each term of the function.

- **Determining Vertical Asymptotes and Holes of a Rational Function:** Determine the value(s) of x that are not in the domain of the function (i.e. set the denominator equal to zero). We know that the graph of $f(x)$ will either have a hole or a vertical asymptote at each of those value(s) of x. We have two methods to determine what is occurring at each value of x:
  - **Option 1:** Evaluate the limit of the function at those particular value(s) of x:
    - If the limit of the function exists (i.e. is a finite number) for that particular value of x, then there is a hole at that value of x.
    - If the limit of the function does not exist for that particular value of x, then there is a vertical asymptote at that value of x.
  - **Option 2:** Factor the numerator and denominator:
    - If the factor divides completely from the denominator, then we have a hole at that x-value.
    - If the factor remains in the denominator after dividing common factors, we have a vertical asymptote at that value of x.

1. Evaluate the following limits:

   (a) $\lim_{x\to-\infty}(-7x^6+2x^3-5x+25)$

   (b) $\lim_{x\to\infty}(4x^2-7x^9+5x^3-9)$

2. Determine the end behavior of $f(x)=-3x^2-Ax^3-9$. where A is a real number such that $A<-3$.

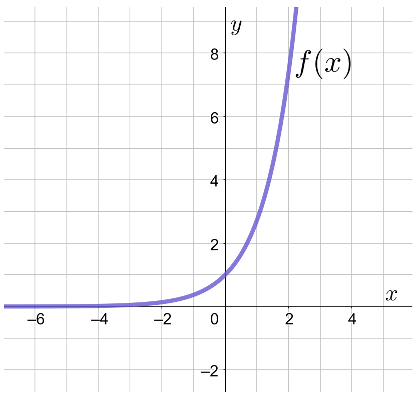
3. Evaluate the following limits algebraically.

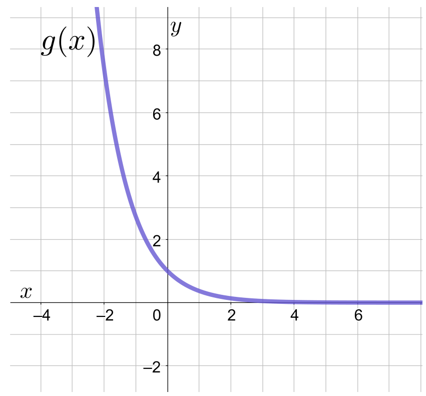
   (a) $\lim_{x\to\infty}\dfrac{4x^7-3x^2+5x}{200-9x^2-5x^8}$

   (b) $\lim_{x\to-\infty}\dfrac{2x^6-5x}{4-5x^3}$

(c) $\lim_{x \to -\infty} \left(\frac{3x}{4x^2 - 7} + \frac{5x^3}{Bx^2 - 15x^3} \right)$ where B is a real number such that $B > 0$

4. Use the graphs of $f(x)$ and $g(x)$ below to determine the following limits.





(a) $\lim_{x \to -\infty} e^x$

(b) $\lim_{x \to \infty} e^x$

(c) $\lim_{x \to -\infty} e^{-x}$

(d) $\lim_{x \to \infty} e^{-x}$

5. Evaluate the following limit algebraically.

$$\lim_{x \to -\infty} \frac{4e^{-3x} + 3e^x + 10}{3e^x + 2e^{-4x}}$$

6. Determine all horizontal asymptotes of $f(x) = \dfrac{5e^{-2x} + 4e^{4x}}{-2e^{5x} - 4e^{-2x}}$

7. Determine all horizontal and vertical asymptote(s) and hole(s) for $f(x) = \dfrac{(x-3)(x+1)}{x^2(x-4)(x+1)}$. For each vertical asymptote, use limit notation to describe the behavior of the function near the vertical asymptote.

Section 1.4

- A function f is **continuous** at the point $x = c$ if all of the following are true:

  I. $f(c)$ is defined

  II. $\lim_{x \to c} f(x)$ exists

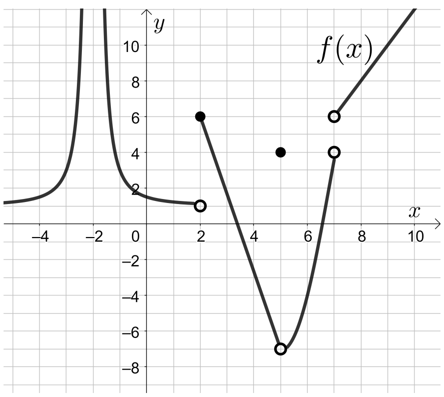
  III. $\lim_{x \to c} f(x) = f(c)$

  If one or more of the conditions are not met, we say f is **discontinuous** at $x = c$.

- Polynomials, exponential functions, logarithmic functions, rational functions, power functions, and combinations of these are continuous on their domain. Thus, to find where these functions are **continuous** we only need to determine the **domain** of the function.

- To determine where a piece-wise defined function is continuous, we must compare the domain restrictions for each rule with the x-values defined for that rule AND check the definition of continuity at each cut-off number.

8. Given the graph of $f(x)$ below, for what value(s) of x is the function discontinuous? What is the first condition to fail?



9. On what interval is each function continuous? Give your answer using interval notation.

   (a) $f(x) = 4x^2 - 7x + 2$

   (b) $g(x) = \dfrac{2x - 7}{x^2 + 4x - 21}$

(c) $v(t) = \dfrac{\sqrt[7]{t-3}}{e^{\frac{t-2}{t}}}$

(d) $h(x) = \dfrac{\sqrt{x-1}}{\ln(-2x+10)}$

(e) $k(x) = \begin{cases} x^2+5 & \text{if } x \leq -5 \\ \dfrac{x-2}{x+4} & \text{if } -5 < x < 0 \end{cases}$

10. Let $g(x) = \dfrac{x^2 - 3x - 10}{2x^2 - 11x + 5}$

(a) Is $g(x)$ continuous at $x = 3$? Use the definition of continuity to justify your answer.

(b) Find all value(s) of x for which $g(x)$ is discontinuous.

11. Find the value(s) of k that makes $f(x)$ continuous at $x = -3$.

$$f(x) = \begin{cases} kx^2 - 9 & \text{if } x < -3 \\ x^2 - 15 & \text{if } -3 \leq x \leq 5 \\ \log(x-4) + 10 & \text{if } x > 5 \end{cases}$$