Math 308: Week-in-Review 6 (3.4-3.6)

1. Use the Method of Reduction of Order to find a second solution to the following differential equation

$$x^2y'' + 3xy' + y = 0, \quad x > 0, \quad y_1(x) = x^{-1}$$

Suppose $y_2(x) = u(x)y_1 = u(x)x^{-1}$ is a second solution.

Plug into ODE: $y_2' = u'x^{-1} - ux^{-2}$

$$y_2'' = u''x^{-1} - u'x^{-2} - u'x^{-2} + 2ux^{-3}$$
$$= u''x^{-1} - 2u'x^{-2} + 2ux^{-3}$$

$$x^2y_2'' + 3xy_2' + y_2 = u''x - 2u' + 2\cancel{u}x^{-1} + 3u' - 3\cancel{u}x^{-1} + u\cancel{x^{-1}}$$
$$= u''x + u' = 0$$

Solve for u: $u''x + u' = 0$. Set $w = u'$.

$$\underbrace{xw' + w}_{\frac{d}{dx}[xw]} = 0$$
$$\frac{d}{dx}[xw] = 0 \Rightarrow xw = c \Rightarrow w = \frac{c}{x}$$

But $w = u' \Rightarrow u' = \frac{c}{x} \Rightarrow u(x) = c\int \frac{1}{x} = c\ln x$

* let $C=1$ * $u(x) = \ln(x)$

$$\boxed{y_2(x) = x^{-1}\ln(x)}$$

2. Suppose you were to use the Method of Undetermined Coefficients to solve the following differential equations. Write out the assumed form of the particular solution, but do not carry out the calculations of the undetermined coefficients.

(a) $y'' + 4y = x^2 - 2x + 1$

Find $y_c(x)$:

Solve $y'' + 4y = 0$:

$\lambda^2 + 4 = 0 \Rightarrow \lambda^2 = -4 \Rightarrow \lambda = \pm 2i$

$\Rightarrow y_c(x) = c_1 \cos(2x) + c_2 \sin(2x)$

Choose $\boxed{y_p(x) = Ax^2 + Bx + C}$

(b) $y'' + 4y = x \sin x$

Find $y_c(x)$: From 2(a) $y_c(x) = c_1 \cos(2x) + c_2 \sin(2x)$

Choose $\boxed{y_p(x) = (Ax + B)\sin(x) + (Cx + D)\cos(x)}$

(c) $y'' + 2y' - 3y = x^2 e^x$

Find $y_c(x)$: $\lambda^2 + 2\lambda - 3 = 0 \Rightarrow (\lambda + 3)(\lambda - 1) = 0$

$\lambda_1 = -3,\ \lambda_2 = 1 \Rightarrow y_c(x) = c_1 e^{-3x} + c_2 e^{x}$

Choose $\boxed{y_p(x) = x\,(Ax^2 + Bx + C)\,e^{x}}$

since e^x is a homogeneous solution, multiply by x here

3. Suppose you were to use the Method of Undetermined Coefficients to solve the following differential equations. Write out the assumed form of the particular solution, but do not carry out the calculations of the undetermined coefficients.

(a) $y'' - 2y' + 5y = xe^x \cos(2x) + e^x \sin(2x)$

Find $y_c(x)$: $\lambda^2 - 2\lambda + 5 = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{2^2 - 4\cdot 5}}{2}$

$= 1 \pm 2i$

$y_c(x) = C_1 e^x \cos(2x) + C_2 e^x \sin(2x)$

Choose $\boxed{y_p(x) = x(Ax+B)e^x \cos(2x) + x(Cx+D)e^x \sin(2x)}$

(b) $y'' + 4y = x\sin x + \cos(2x)$

Find $y_c(x)$: $y_c(x) = C_1 \cos(2x) + C_2 \sin(2x)$

Choose $\boxed{y_p(x) = (Ax+B)\cos(x) + (Cx+D)\sin(x) + xE\cos(2x) + xF\sin(2x)}$

4. Solve the following equations using the Method of Undetermined Coefficients. If any initial value is given, then solve the initial value problem. If no initial value is given, then find the general solution. Find an explicit solution if possible.

   (a) $f'' - 7f' + 12f = 2e^{5t}$, $f(0) = 0$, $f'(0) = -1$.

Find $f_c(t)$: $\lambda^2 - 7\lambda + 12 = 0 \Rightarrow (\lambda-3)(\lambda-4) = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = 4$

$f_c(t) = c_1 e^{3t} + c_2 e^{4t}$

$f_p(t) = Ae^{5t}$, $f_p' = 5Ae^{5t}$, $f_p'' = 25Ae^{5t}$

$f_p'' - 7f_p' + 12f_p = Ae^{5t}[25 - 7\cdot 5 + 12] = 2Ae^{5t} = 2e^{5t}$

$A = 1$, $\boxed{f_p(t) = e^{5t}}$

General solution: $f(t) = c_1 e^{3t} + c_2 e^{4t} + e^{5t}$

$f'(t) = 3c_1 e^{3t} + 4c_2 e^{4t} + 5e^{5t}$

$\left\{\begin{array}{l} f(0) = c_1 + c_2 + 1 = 0 \\ f'(0) = 3c_1 + 4c_2 + 5 = -1 \end{array}\right.$

$c_1 = -1 - c_2 \Rightarrow 3(-1 - c_2) + 4c_2 = -6 \Rightarrow c_2 = 3 - 6 = -3$

$c_1 = 2$

$$\boxed{f(t) = 2e^{3t} - 3e^{4t} + e^{5t}}$$

5.

$$g'' + 2g' + 2g = 2t, \quad g(0) = 0, \quad g'(0) = 1.$$

Find $y_c(t)$: $\lambda^2+2\lambda+2=0 \Rightarrow \lambda = \dfrac{-2\pm\sqrt{2^2-4\cdot 2}}{2}$

$= \dfrac{-2\pm 2i}{2} = -1\pm i$

$g_c(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$

$g_p(t) = At+B, \quad g_p' = A, \quad g_p'' = 0$

$g_p'' + 2g_p' + 2g_p = 0 + 2A + 2[At+B] = 2At + (2A+2B)$

$= 2t$

$2A = 2 \Rightarrow A = 1, \quad 2A+2B = 0 \Rightarrow 2B = -2A$

$2B = -2 \Rightarrow B = -1$

$g_p(t) = t-1$

General solution: $g(t) = g_c(t) + g_p(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) + t - 1$

$g'(t) = -c_1 e^{-t}\cos(t) - c_1 e^{-t}\sin(t) - c_2 e^{-t}\sin(t) + c_2 e^{-t}\cos(t) + 1$

$g(0) = c_1 - 1 = 0 \Rightarrow c_1 = 1$

$g'(0) = -c_1 + c_2 + 1 = 1 \Rightarrow c_2 = 1$

$$\boxed{g(t) = e^{-t}\cos(t) + e^{-t}\sin(t) + t - 1}$$

6.

$$u'' + 2u' + u = 2e^{-t}$$

Find $u_c(t)$: $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda+1)^2 = 0 \Rightarrow \lambda = -1$ * repeated *

$u_c(t) = c_1 e^{-t} + c_2 t e^{-t}$

$u_p(t) = A t^2 e^{-t}$

$u_p' = 2Ate^{-t} - At^2e^{-t}$, $u_p'' = 2Ae^{-t} - 2Ate^{-t} - 2Ate^{-t} + At^2e^{-t}$

$u_p'' + 2u_p' + u_p = [2Ae^{-t} - \cancel{4Ate^{-t}} + \cancel{At^2e^{-t}}] + 2[\cancel{2Ate^{-t}} - \cancel{At^2e^{-t}}] + \cancel{At^2e^{-t}} = 2Ae^{-t} = 2e^{-t} \Rightarrow A = 1$

$u_p(t) = t^2 e^{-t}$

$\boxed{u(t) = c_1 e^{-t} + c_2 t e^{-t} + t^2 e^{-t}}$ General solution

7. Solve the following equations using the Method of Variation of Parameters. If any initial value is given, then solve the initial value problem. If no initial value is given, then find the general solution. Find an explicit solution if possible.

(a)

$$u'' + 2u' + u = 2e^{-t}$$

$u_c(t) = c_1 e^{-t} + c_2 t e^{-t}$, $y_1 = e^{-t}$, $y_2 = te^{-t}$

$u(t) = u_c(t) + u_p(t)$, $u_p(t) = u_1 y_1 + u_2 y_2 = u_1 e^{-t} + u_2 t e^{-t}$

$u_1 = \int \frac{-y_2 r}{W} dt = \int \frac{-t\cancel{e^{-t}} \cdot 2\cancel{e^{-t}}}{\cancel{e^{-2t}}}$

$= \int -2t\,dt$

$= -t^2$

where $W = y_1 y_2' - y_1' y_2$

$= e^{-t}(e^{-t} - te^{-t})$

$- te^{-t}(-e^{-t})$

$= e^{-2t} - \cancel{te^{-2t}} + \cancel{te^{-2t}}$

$= e^{-2t}$

$u_2 = \int \frac{y_1 r}{W} dt = \int \frac{\cancel{e^{-t}} \cdot 2\cancel{e^{-t}}}{\cancel{e^{-2t}}} dt = 2t$

$u_p(t) = u_1 y_1 + u_2 y_2 = -t^2 e^{-t} + 2t^2 e^{-t} = t^2 e^{-t}$

$$\boxed{u(t) = c_1 e^{-t} + c_2 t e^{-t} + t^2 e^{-t}}$$

(b) Solve the initial value problem

$$3y'' + 4y' + y = e^{-t}\sin(t), \quad y(0) = 1, \; y'(0) = 0.$$

Find $y_c(t)$: $3\lambda^2 + 4\lambda + 1 = 0 \Rightarrow \lambda = \dfrac{-4 \pm \sqrt{4^2 - 4 \cdot 3 \cdot 1}}{6}$

$= -\frac{4}{6} \pm \frac{\sqrt{4}}{6} = -\frac{2}{3} \pm \frac{1}{3}$

$= -1, -1/3$

$y_c(t) = c_1 e^{-t} + c_2 e^{-t/3}$

$y_1(t) = e^{-t}, \quad y_2 = e^{-t/3}$

$$W = y_1 y_2' - y_1' y_2 = (e^{-t})(-\tfrac{1}{3}e^{-t/3}) - (-e^{-t})(e^{-t/3}) = -\tfrac{1}{3}e^{-4/3 t} + e^{-4/3 t} = \tfrac{2}{3}e^{-4/3 t}$$

$y_p(t) = u_1 y_1 + u_2 y_2 = u_1 e^{-t} + u_2 e^{-t/3}$

* standard form * $y'' + \frac{4}{3}y' + \frac{1}{3}y = \overbrace{\frac{1}{3}e^{-t}\sin(t)}^{r(t)}$

$$u_1 = \int -\frac{\cancel{e^{-t/3}} \cdot \frac{1}{3}\cancel{e^{-t}}\sin(t)}{\frac{2}{3}\cancel{e^{-4/3}}}\,dt = \frac{\cancel{3}}{2}\cdot\frac{1}{\cancel{3}}\int -\sin(t)\,dt = \frac{1}{2}\cos(t)$$

* by parts *

$$u_2 = \int \frac{e^{-t} \cdot \frac{1}{3}e^{-t}\sin(t)}{\frac{2}{3}e^{-4/3}}\,dt = \frac{1}{2}\int e^{-2/3 t}\sin(t)\,dt = -\frac{3}{13}e^{-2/3 t}\sin(t) - \frac{9}{26}e^{-2/3 t}\cos(t)$$

$$y_p(t) = \frac{1}{2}e^{-t}\cos(t) - \frac{3}{13}e^{-t}\sin(t) - \frac{9}{26}e^{-t}\cos(t)$$

and $y(t) = c_1 e^{-t} + c_2 e^{-t/3} + \frac{1}{2}e^{-t}\cos(t) - \frac{3}{13}e^{-t}\sin(t) - \frac{9}{26}e^{-t}\cos(t)$

* find c_1, c_2: $c_1 = -1$, $c_2 = \frac{24}{13}$

$$\boxed{y(t) = -e^{-t} + \frac{24}{13}e^{-t/3} + \frac{e^{-t}}{13}\left(2\cos(t) - 3\sin(t)\right)}$$

8. Given the complementary solution $y_c(t) = \underline{C_1 t + C_2 t^{-1}}$, use Variation of Parameters to find a particular of the differential equation

$$t^2 y'' + ty' - 2y = t^2, \quad t > 0.$$

$$y_1 = t, \quad y_2 = t^{-1} \Rightarrow W(t) = y_1 y_2' - y_1' y_2 = t \cdot (-t^{-2}) - 1 \cdot t^{-1}$$
$$= -t^{-1} - t^{-1}$$
$$= -2t^{-1}$$

$$y_p = u_1 y_1 + u_2 y_2 \text{ where } u_1 = \int -\frac{y_2 r(t)}{W(t)} dt, \quad u_2 = \int \frac{y_1 r(t)}{W(t)} dt$$

* $r(t)$ is the right hand side of the equation in <u>STANDARD FORM</u>

$$y'' + \frac{1}{t} y' - \frac{2}{t^2} y = 1, \quad t > 0 \Rightarrow r(t) = 1$$

$$y_p(t) = u_1 y_1 + u_2 y_2, \quad u_1 = \int -\frac{y_2 r}{W} dt = \int -\frac{\cancel{t^{-1}} \cdot 1}{-2\cancel{t^{-1}}} dt = \frac{1}{2} t$$

$$u_2 = \int \frac{y_1 r}{W} dt = \int \frac{t \cdot 1}{-2t^{-1}} = -\frac{1}{2} \int t^2 dt = -\frac{1}{6} t^3$$

$$y_p(t) = \frac{1}{2} t^2 - \frac{1}{6} t^2 = \frac{1}{3} t^2$$

$$\boxed{y_p(t) = \frac{1}{3} t^2}$$

9. Find two linearly independent solutions of $t^2y'' - 2y = 0$ of the form $y(t) = t^r$. Using these solutions, find the general solution of $t^2y'' - 2y = t^2$. , $t>0$

Plug in $y(t) = t^r$ into the equation:

$* \; y'' - \frac{2}{t^2}y = 1 \; *$

$\boxed{r(t) = 1}$

$$y' = rt^{r-1}, \quad y'' = r(r-1)t^{r-2}$$

$$t^2y'' - 2y = [r(r-1) - 2]t^r = [r^2 - r - 2]t^r = 0$$

$$\Rightarrow r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0$$

$$r = 2, \; r = -1$$

$$y_1(t) = t^2, \quad y_2(t) = t^{-1} \Rightarrow W(t) = y_1y_2' - y_1'y_2 = (t^2)(-t^{-2}) - (2t)(t^{-1})$$

$$= -1 - 2 = -3$$

$$y_p(t) = u_1y_1 + u_2y_2$$

$$= \frac{1}{3}t^2\ln t - \frac{1}{9}t^2$$

$$u_1 = \int -\frac{t^{-1} \cdot 1\,dt}{-3} = \frac{1}{3}\ln t$$

$$u_2 = \int \frac{t^2 \cdot 1}{-3} = -\frac{1}{9}t^3$$

↳ this term is proportional to $y_1(t) \Rightarrow$ homogeneous

$$y_p(t) = \frac{1}{3}t^2\ln t$$

$$\boxed{y(t) = c_1t^2 + c_2t^{-1} + \frac{1}{3}t^2\ln t}$$

10. One solution of $4t^2y'' + 4ty' + (16t^2 - 1)y = 0$, $t > 0$ is $y(t) = t^{-1/2}\cos(2t)$. Find the general solution of $4t^2y'' + 4ty' + (16t^2 - 1)y = 16t^{3/2}$.

$$* \; y'' + \frac{1}{t}y' + \left(4 - \frac{1}{4t^2}\right)y = 4t^{-1/2} \; *$$ standard form equation

1. $y_1(t) = t^{-1/2}\cos(2t)$. Find $y_2(t)$.

$$y_2(t) = y_1(t)\int \frac{e^{\int -p(t)dt}}{y_1^2(t)}\,dt = t^{-1/2}\cos(2t)\int \frac{e^{\int -1/t\,dt}}{t^{-1}\cos^2(2t)}\,dt$$

$$= t^{-1/2}\cos(2t)\int \frac{\cancel{t^{-1}}}{\cancel{t^{-1}}\cos^2(2t)}\,dt = t^{-1/2}\cos(2t)\int \sec^2(2t)\,dt$$

$$= t^{-1/2}\cos(2t)\tan(2t)\cdot\frac{1}{2} = \frac{1}{2}t^{-1/2}\cancel{\cos(2t)}\cdot\frac{\sin(2t)}{\cancel{\cos(2t)}}$$

$$= \frac{1}{2}t^{-1/2}\sin(2t)$$

$$\boxed{y_2(t) = t^{-1/2}\sin(2t)}$$

$*$ Variation of parameters: $W(t) = y_1y_2' - y_1'y_2$
$= 2t^{-1}$

$$u_1 = \int -\frac{\cancel{t^{-1/2}}\sin(2t)\cdot 4\cancel{t^{-1/2}}}{2\cancel{t^{-1}}}$$

$$= 2\int -\sin(2t)\,dt = \cos(2t)$$

$$u_2 = \int \frac{\cancel{t^{-1/2}}\cos(2t)\cdot 4\cancel{t^{-1/2}}}{2\cancel{t^{-1}}}\,dt$$

$$= \sin(2t)$$

$$y_p(t) = t^{-1/2}\cos^2(2t) + t^{-1/2}\sin^2(2t)$$

$$= t^{-1/2}\Big[\underbrace{\cos^2(2t) + \sin^2(2t)}_{1}\Big]$$

$$= t^{-1/2}$$

$$\boxed{y(t) = t^{-1/2}\left[\cos(2t) + \sin(2t) + 1\right]}$$