Week 1

Monday, August 26, 2024 7:27 PM

Math 151
Week-In-Review 1

Appendix J.1 and J.2
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Problem Statements

1. (a) Find a vector from the point $A(-2, 3)$ to the point $B(5, -1)$.

$$\vec{AB} = \langle 5-(-2), -1-3 \rangle = \langle 7, -4 \rangle$$

(b) What is the magnitude of the vector?

$$|\vec{AB}| = \sqrt{7^2 + (-4)^2} = \sqrt{49+16} = \sqrt{65}$$

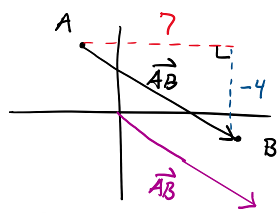
(c) Find $\overrightarrow{BA}$.

$$\vec{BA} = \langle -2-5, 3-(-1) \rangle = \langle -7, 4 \rangle$$

(d) What is $|\overrightarrow{BA}|$?

$$|\vec{BA}| = \sqrt{(-7)^2 + 4^2} = \sqrt{49+16} = \sqrt{65}$$

(e) Plot the points A and B, and sketch vector $\overrightarrow{AB}$.



(f) Sketch the position vector of $\overrightarrow{AB}$ in the same coordinate plane.

$\vec{a}$

2. Given vectors $\boldsymbol{a} = \langle 2, -5 \rangle$, $\boldsymbol{b} = 4\boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{c} = 6\boldsymbol{j}$, and $\boldsymbol{d} = \langle -7, 1 \rangle$, evaluate/find the following, if possible.

$\vec{b} = \langle 4, 2 \rangle \qquad \vec{c} = \langle 0, 6 \rangle$

(a) $3\boldsymbol{a} - 4\boldsymbol{b}$

$$3\langle 2,-5\rangle - 4\langle 4,2\rangle = \langle 6, -15\rangle - \langle 16, 8\rangle = \langle -10, -23\rangle$$

(b) $\boldsymbol{d} \cdot \boldsymbol{c}$

$$\langle -7,1\rangle \cdot \langle 0,6\rangle = -7(0) + 1(6) = 0+6 = \boxed{6}$$

(c) $(3\boldsymbol{a}) \cdot \boldsymbol{d}$

$$(3\langle 2,-5\rangle)\cdot\langle -7,1\rangle = \langle 6,-15\rangle \cdot \langle -7,1\rangle = 6(-7) + -15(1) = \boxed{-57}$$

(d) $3(\boldsymbol{a} \cdot \boldsymbol{d})$

$$3(\langle 2,-5\rangle\langle -7,1\rangle) = 3\,(2(-7) + -5(1)) = 3(-19) = \boxed{-57}$$

(e) $\boldsymbol{b} \cdot \boldsymbol{c} \cdot \boldsymbol{d}$

Not Possible

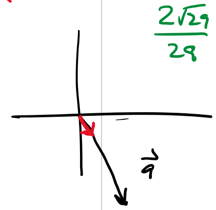
$$(\langle 4,2\rangle \cdot \langle 0,6\rangle)\cdot\langle -7,1\rangle = (4(0) + 2(6))\cdot\langle -7,1\rangle = \cancel{12 \cdot \langle -7,1\rangle}$$

$$\frac{2}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}}$$

$$\frac{2\sqrt{29}}{29}$$

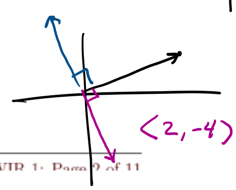
(f) A unit vector in the direction of $\boldsymbol{a}$ $\qquad \vec{a} = \langle 2, -5\rangle$

$$\vec{u} = \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \boxed{\frac{\langle 2,-5\rangle}{\sqrt{29}}} = \left\langle \frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}} \right\rangle \qquad |\vec{a}| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$$



(g) The orthogonal complement of $\boldsymbol{b}$ $= \langle 4, 2\rangle$

$$\vec{b}^{\perp} = \langle -2, 4 \rangle$$



(h) A unit vector perpendicular to $\boldsymbol{d}$ $= \langle -7, 1 \rangle$

$\vec{d}^{\perp} = \langle -1, -7 \rangle$ Another perpendicular vector $\langle 1, 7 \rangle = \vec{v}$

$$\hat{d}^{\perp} = \frac{\langle -1,-7\rangle}{\sqrt{(-1)^2+(-7)^2}} = \frac{\langle -1,-7\rangle}{\sqrt{50}}$$

$$\hat{v} = \frac{\langle 1,7\rangle}{\sqrt{1^2+7^2}} = \frac{\langle 1,7\rangle}{\sqrt{50}}$$

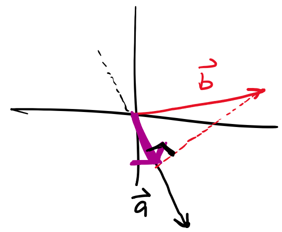
(i) The scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$

$$\text{comp}_{\vec a}\vec b = \frac{\vec a \cdot \vec b}{|\vec a|} = \frac{\langle 2,-5\rangle \cdot \langle 4,2\rangle}{|\langle 2,-5\rangle|} = \frac{2(4) + -5(2)}{\sqrt{2^2+(-5)^2}} = \frac{8-10}{\sqrt{29}} = \boxed{\frac{-2}{\sqrt{29}}}$$

(j) The vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$

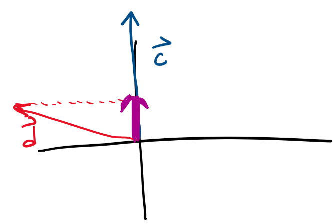
$$\text{proj}_{\vec a}\vec b = \frac{\vec a \cdot \vec b}{|\vec a|}\left(\frac{\vec a}{|\vec a|}\right) = \frac{-2}{\sqrt{29}}\,\frac{\langle 2,-5\rangle}{\sqrt{29}} = \boxed{\frac{-2}{29}\langle 2,-5\rangle}$$

unit vector



(k) $\text{comp}_{\boldsymbol{c}}\boldsymbol{d}$

$$= \frac{\vec c \cdot \vec d}{|\vec c|} = \frac{\langle 0,6\rangle \cdot \langle -7,1\rangle}{\langle 0,6\rangle} = \frac{0(-7) + 6(1)}{\sqrt{0^2+6^2}} = \frac{6}{\sqrt{36}} = \frac{6}{6} = \boxed{1}$$



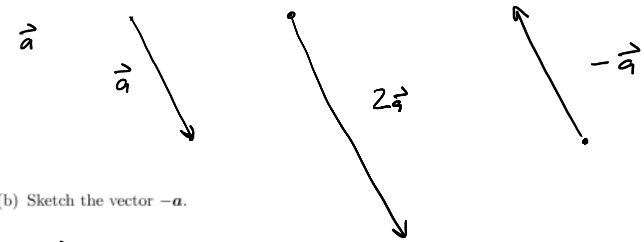
(l) $\text{proj}_{\boldsymbol{c}}\boldsymbol{d}$

$$= \frac{\vec c \cdot \vec d}{|\vec c|}\left(\frac{\vec c}{|\vec c|}\right) = 1\,\frac{\langle 0,6\rangle}{6} = 1\langle 0,1\rangle = \boxed{\langle 0,1\rangle}$$

3. Consider the vectors $\boldsymbol{a} = \langle 2, -5 \rangle$ and $\boldsymbol{b} = 4\boldsymbol{i} + 2\boldsymbol{j}$ from example 2.

(a) Sketch the vector $2\boldsymbol{a}$.

$2\vec{a} = \langle 4, -10 \rangle$



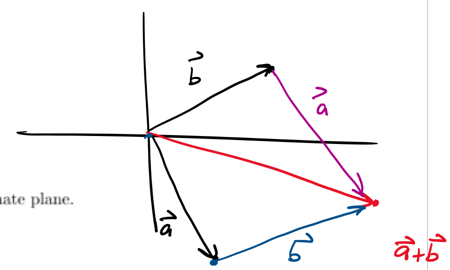
(b) Sketch the vector $-\boldsymbol{a}$.

$-\vec{a} = \langle -2, 5 \rangle$

(c) Sketch the position vectors of $\boldsymbol{a}$ and $\boldsymbol{b}$ in a coordinate plane.

$\vec{a} + \vec{b} = \langle 2 + 4, -5 + 2 \rangle$

$= \langle 6, -3 \rangle$



(d) Sketch the vector $\boldsymbol{a} + \boldsymbol{b}$ in the same coordinate plane.

4. A drone flies through the air in the direction N 30° W at a (still-air) speed of 5 m/s. A wind is blowing the ~~kite~~ drone N 45° E at a speed of 10 m/s.

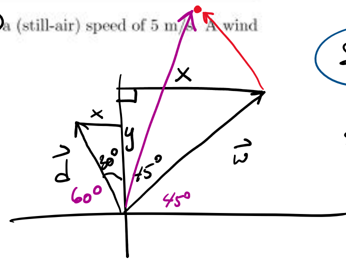
(a) Determine the drone's resulting velocity.

$\vec{d} + \vec{w}$

$\vec{w} = \langle 10 \sin(45), \ 10 \cos(45) \rangle$

$\vec{d} = \langle -5 \sin(30), \ 5 \cos(30) \rangle$

$\vec{w} = \langle 10\left(\frac{\sqrt{2}}{2}\right), 10\left(\frac{\sqrt{2}}{2}\right) \rangle = \langle 5\sqrt{2}, 5\sqrt{2} \rangle$



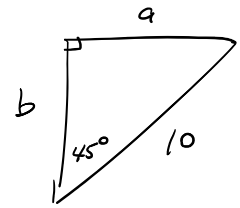
$S\frac{O}{H}\ C\frac{A}{H}\ T\frac{O}{A}$

$\sin(\theta) = \frac{x}{10}$

$x = 10 \sin(45)$

$\cos(\theta) = \frac{y}{10}$

$y = 10 \cos(45)$



$\sin(45) = \frac{a}{10}$

(b) Determine the drone's speed through the air.

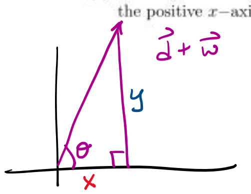
$\vec{d} = \langle -5\left(\frac{1}{2}\right), \ 5\left(\frac{\sqrt{3}}{2}\right) \rangle = \langle \frac{-5}{2}, \frac{5\sqrt{3}}{2} \rangle$

(a) $\boxed{\vec{d} + \vec{w} = \langle \underbrace{5\sqrt{2} - \frac{5}{2}}_{x}, \ \underbrace{5\sqrt{2} + \frac{5\sqrt{3}}{2}}_{y} \rangle}$

Speed:

(b) $|\vec{d} + \vec{w}| = \sqrt{\left(5\sqrt{2} - \frac{5}{2}\right)^2 + \left(5\sqrt{2} + \frac{5\sqrt{3}}{2}\right)^2}$

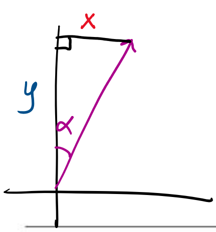
(c) Determine the drone's direction of motion as an angle from the positive x−axis, assuming the positive x−axis represents East.



$\tan(\theta) = \frac{y}{x}$

$\boxed{\theta = \arctan\left(\frac{5\sqrt{2} + \frac{5\sqrt{3}}{2}}{5\sqrt{2} - \frac{5}{2}}\right)}$

(d) Determine the drone's direction of motion as a bearing.

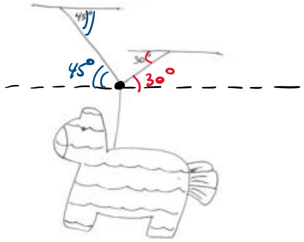


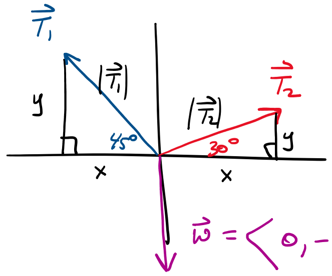
$\tan(\alpha) = \frac{x}{y}$

N α° E

$\alpha = \arctan\left(\frac{5\sqrt{2} - \frac{5}{2}}{5\sqrt{2} + \frac{5\sqrt{3}}{2}}\right)$

5. Suppose a 30 pound piñata is supported by two ropes, as in the crudely drawn picture below. Determine the magnitude of the tension in each rope.





$\vec{w} = \langle 0, -30 \rangle$

$$\vec{T}_1 = \langle -|\vec{T}_1| \cos(45), |\vec{T}_1| \sin(45) \rangle$$
$$= \langle -|\vec{T}_1| \cdot \tfrac{\sqrt{2}}{2}, |\vec{T}_1| \cdot \tfrac{\sqrt{2}}{2} \rangle$$

$$\vec{T}_2 = \langle |\vec{T}_2| \cos(30), |\vec{T}_2| \sin(30) \rangle$$
$$= \langle |\vec{T}_2| \cdot \tfrac{\sqrt{3}}{2}, |\vec{T}_2| \cdot \tfrac{1}{2} \rangle$$

$$\vec{T}_1 + \vec{T}_2 + \vec{w} = \langle \underline{-|\vec{T}_1| \tfrac{\sqrt{2}}{2} + |\vec{T}_2| \tfrac{\sqrt{3}}{2}}, \underline{|\vec{T}_1| \tfrac{\sqrt{2}}{2} + |\vec{T}_2| \tfrac{1}{2} - 30} \rangle$$
$$= \langle \underline{0}, \underline{0} \rangle$$

$$-|\vec{T}_1| \tfrac{\sqrt{2}}{2} + |\vec{T}_2| \tfrac{\sqrt{3}}{2} = 0$$
$$|\vec{T}_2| \tfrac{\sqrt{3}}{\cancel{2}} = |\vec{T}_1| \tfrac{\sqrt{2}}{\cancel{2}}$$
$$|\vec{T}_2| = |\vec{T}_1| \cdot \tfrac{\sqrt{2}}{\sqrt{3}}$$

$$|\vec{T}_1| \tfrac{\sqrt{2}}{2} + |\vec{T}_2| \left(\tfrac{1}{2}\right) - 30 = 0$$
$$|\vec{T}_1| \tfrac{\sqrt{2}}{2} + |\vec{T}_1| \tfrac{\sqrt{2}}{\sqrt{3}} \cdot \tfrac{1}{2} = 30$$
$$|\vec{T}_1| \left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2\sqrt{3}} \right) = 30$$

$$\boxed{|\vec{T}_1| = \frac{30}{\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2\sqrt{3}} \right)}}$$

$$\boxed{|\vec{T}_2| = \frac{30}{\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2\sqrt{3}} \right)} \cdot \frac{\sqrt{2}}{\sqrt{3}}}$$

6. Consider the vectors $\boldsymbol{a} = \langle 1, 2 \rangle$, $\boldsymbol{b} = \langle -4, 2 \rangle$, $\boldsymbol{c} = \langle -3, -6 \rangle$, and $\boldsymbol{d} = \langle 2, -5 \rangle$.

(a) Determine if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, perpendicular, or neither.

$\perp$: $\langle 1,2 \rangle \cdot \langle -4,2 \rangle = 1(-4) + 2(2) = -4 + 4 = 0$ ✓

$\vec{a} \cdot \vec{b} = 0$?

$\boxed{\vec{a} \text{ and } \vec{b} \text{ are perpendicular}}$

(b) Determine if $\boldsymbol{a}$ and $\boldsymbol{c}$ are parallel, perpendicular, or neither.

$\perp$: $\langle 1,2 \rangle \cdot \langle -3,-6 \rangle = 1(-3) + 2(-6) = -15$ X (Not perpendicular)

$\vec{a} \cdot \vec{c}$

Parallel: $\vec{a} = m\vec{c}$? $\langle 1,2 \rangle = m \langle -3,-6 \rangle$

$1 = -3m$ $\quad$ $2 = -6m$

$m = \frac{-1}{3}$ $\quad$ $m = \frac{-2}{6} = \frac{-1}{3}$ ✓

$\boxed{\vec{a} \text{ and } \vec{c} \text{ are parallel}}$

(c) Determine if $\boldsymbol{a}$ and $\boldsymbol{d}$ are parallel, perpendicular, or neither.

$\perp$: $\vec{a} \cdot \vec{d} = 0$? $\langle 1,2 \rangle \cdot \langle 2,-5 \rangle = 1(2) + 2(-5) = -8$ X Not perpendicular

Parallel: $\vec{a} = n\vec{d}$

$\langle 1,2 \rangle = n \langle 2,-5 \rangle$ Not parallel

$1 = 2n$ $\quad$ $2 = -5n$

$n = \frac{1}{2}$ X $n = \frac{-2}{5}$

$\boxed{\vec{a} \text{ and } \vec{d} \text{ are neither parallel nor perpendicular}}$

7. A constant force with a vector representation $\boldsymbol{F} = 11\boldsymbol{i} + 15\boldsymbol{j}$ moves an object along a straight line from the point $(-1, 2)$ to $(4, 8)$. Find the work done if the distance is measured in meters and the force is measure in newtons.

$\vec{F} = \langle 11, 15 \rangle$

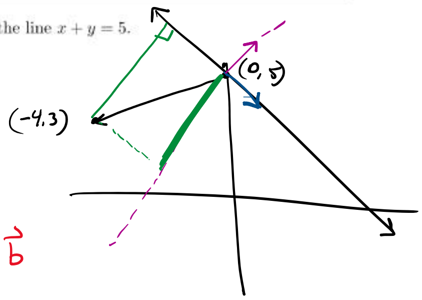
$\vec{d} = \langle 4-(-1),\ 8-2 \rangle = \langle 5, 6 \rangle$

$W = |\vec{F}|\,|\vec{d}|\cos(\theta) = \vec{F}\cdot\vec{d}$

$\vec{F}\cdot\vec{d} = 11(5) + 15(6) = 55 + 90 = 145\ N\cdot m = \boxed{145\ J}$

$y = 5 - x$

8. Find the (shortest) distance from the point $(-4, 3)$ to the line $x + y = 5$.



1. Any vector from line to point.

$\langle -4-0, 3-5 \rangle = \langle -4, -2 \rangle = \vec{b}$

2. Vector in direction of line. Slope vector.

$m = \dfrac{-1}{1} = \dfrac{\Delta y}{\Delta x}$ $\quad \Delta x = 1 \quad \Delta y = -1$

$\vec{m} = \langle 1, -1 \rangle$

$\vec{m} = \langle \Delta x, \Delta y \rangle = \langle 1, -1 \rangle$

3. Vector perpendicular to the line.

$\vec{m}^{\perp} = \langle 1, 1 \rangle = \vec{a}$

4. $$\left|\text{comp}_{\vec{a}} \vec{b}\right| = \left|\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right| = \left|\frac{1(-4) + 1(-2)}{\sqrt{1^2 + 1^2}}\right| = \left|\frac{-6}{\sqrt{2}}\right| = \boxed{\frac{6}{\sqrt{2}}}$$

9. (if time) Find the distance from the point $(1, 5)$ to the line $y = 3x - 8$.

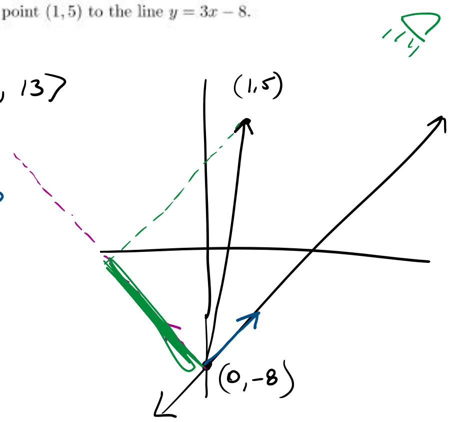
1. $\vec{b} = \langle 1-0,\ 5-(-8)\rangle = \langle 1, 13\rangle$

2. $\vec{m} = \langle \Delta x, \Delta y\rangle = \langle 1, 3\rangle$

3. $\vec{m}^{\perp} = \langle -3, 1\rangle = \vec{a}$

4. $\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

$$= \frac{1(-3) + 13(1)}{\sqrt{(-3)^2 + 1^2}} = \frac{10}{\sqrt{10}} = \boxed{\sqrt{10}}$$



$m = \dfrac{3}{1} = \dfrac{\Delta y}{\Delta x}$

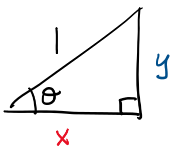
Unit Circle Discussion

Any extra time will be devoted to discussing the unit circle/giving pointers on how to memorize it. We will use the values from the unit circle throughout all of Calculus and your future Math courses. You will be expected to simplify all expressions involving normal unit circle angles on exams.

$\langle 2\cos(45),\ 2\sin(45)\rangle \qquad \langle 2\cdot\frac{\sqrt{2}}{2},\ 2\cdot\frac{\sqrt{2}}{2}\rangle$

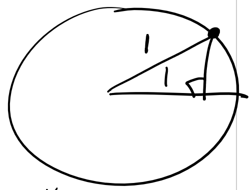
$\langle 1.414,\ 1.414\rangle$

S $\frac{O}{H}$ C $\frac{A}{H}$ T $\frac{O}{A}$



$\sin(\theta) = \frac{y}{1}$

$y = \sin\theta$



$\cos(\theta) = \frac{x}{1}$

$x = \cos(\theta)$

$\cos(90) = 0$

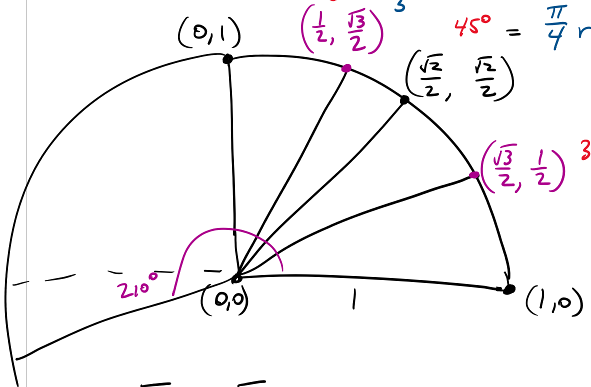
$\sin\left(\frac{\pi}{2}\right) = 1$

$90^\circ = \frac{\pi}{2}$ rad

$60^\circ = \frac{\pi}{3}$ rad

$45^\circ = \frac{\pi}{4}$ rad

$30^\circ = \frac{\pi}{6}$ rad



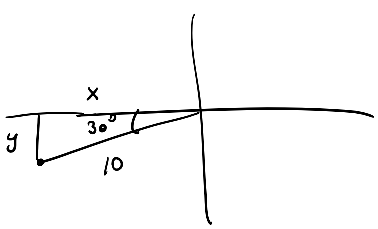
π rad $= 180^\circ$

$45^\circ \cdot \frac{\pi}{180^\circ}$

$\theta = 0^\circ$ or 0 rad

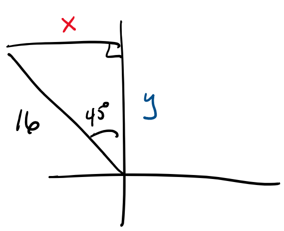
$\cos(0) = 1 \qquad \sin(0) = 0$

$\frac{\sqrt{0}}{2} \qquad \frac{\sqrt{1}}{2} \qquad \frac{\sqrt{2}}{2} \qquad \frac{\sqrt{3}}{2} \qquad 1 = \frac{\sqrt{4}}{2}$



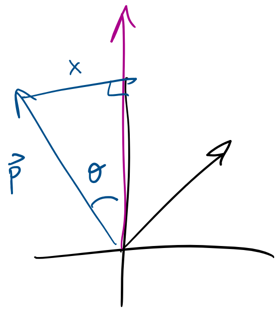
$x = -10\cos(30) = -10\left(\frac{\sqrt{3}}{2}\right)$

$y = -10\sin(30) = -10\left(\frac{1}{2}\right) = -5$



$x = -16\sin(45) = -16\cdot\frac{\sqrt{2}}{2} = -8\sqrt{2}$

$y = 16\cos(45) = 16\cdot\frac{\sqrt{2}}{2} = 8\sqrt{2}$



$\vec{p} = \langle -750\sin(\theta),\ 750\cos(\theta) \rangle$

$\vec{w} = \langle 10\ ,\ 20 \rangle$

$\vec{p} + \vec{w} = \langle 0\ ,\ ??? \rangle$