10. Use the given transformation to evaluate the integral

(a) $\iint_R (x+y)e^{x^2-y^2}\,dA$, where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$ and $x+y=3$.

$\boxed{x+y=u \;/\; x-y=v}$

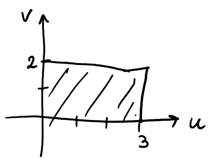
$x^2-y^2 = \underbrace{(x-y)}_{v}\underbrace{(x+y)}_{u}$

$0 \le v \le 2, \quad 0 \le u \le 3$

$x-y=0 \longrightarrow v=0$
$x-y=2 \longrightarrow v=2$
$x+y=0 \longrightarrow u=0$
$x+y=3 \longrightarrow u=3$

$dA = |J|\,du\,dv = \left|-\frac{1}{2}\right| du\,dv = \frac{1}{2}\,du\,dv$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = -\frac{1}{2}$$



$$= \frac{1}{2}\int_0^3\int_0^2 u e^{uv}\,dv\,du = \frac{1}{2}\int_0^3 \cancel{u}\,\frac{1}{\cancel{u}} e^{uv}\Big|_{v=0}^{v=2} du$$

$$= \frac{1}{2}\int_0^3 \left(e^{2u}-1\right) du = \frac{1}{2}\left(\frac{1}{2}e^{2u} - u\right)\Big|_0^3 = \frac{1}{2}\left(\frac{1}{2}e^{6} - \frac{1}{2} - 3\right)$$

$$= \boxed{\frac{1}{4}\left(e^6 - 7\right)}$$

$u = x+y$
$+\; v = x-y$

$u+v = 2x \Rightarrow x = \frac{1}{2}(u+v)$
$u-v = 2y \Rightarrow y = \frac{1}{2}(u-v)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\int_a^b \int_c^d f(x)g(y)\,dy\,dx = \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dy$$

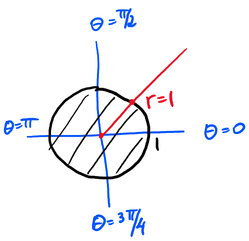
10. Use the given transformation to evaluate the integral

(b) $\iint_R x^2\,dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$,

$$\begin{cases} x = 2r\cos\theta \\ y = 3r\sin\theta \end{cases}$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2\cos\theta & -2r\sin\theta \\ 3\sin\theta & 3r\cos\theta \end{vmatrix} = 6r\cos^2\theta + 6r\sin^2\theta = 6r(\cos^2\theta + \sin^2\theta) = 6r$$

$$\boxed{dA = 6r\,dr\,d\theta}$$



$$9(4r^2\cos^2\theta) + 4(9r^2\sin^2\theta) = 36$$

$$\frac{36r^2\cos^2\theta + 36r^2\sin^2\theta}{36} = \frac{36}{36}$$

$$r^2(\cos^2\theta + \sin^2\theta) = 1 \longrightarrow r = 1$$

origin $\leq r \leq$ circle

$$\boxed{0 \leq r \leq 1}$$

$$\boxed{0 \leq \theta \leq 2\pi}$$

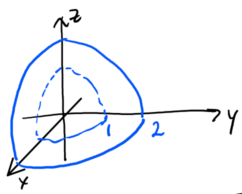
$$\int_0^{2\pi}\int_0^1 4r^2\cos^2\theta\,(6r)\,dr\,d\theta = 24\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\,dr\,d\theta$$

$$= 24\int_0^{2\pi}\cos^2\theta\,d\theta \cdot \int_0^1 r^3\,dr$$

$$= 24\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta \left(\frac{r^4}{4}\right)\Big|_0^1$$

$$= 3\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = \boxed{6\pi}$$

9. Evaluate $\iiint_E xe^{(x^2+y^2+z^2)^2}\,dV$ if the E is the solid that lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in the first octant.



spherical $\begin{cases} x=\rho\cos\theta\sin\varphi \\ y=\rho\sin\theta\sin\varphi \\ z=\rho\cos\varphi \end{cases}$ $\quad dV=\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

$x^2+y^2+z^2=\rho^2$

$x^2+y^2+z^2=1 \longrightarrow \rho^2=1$
$\rho=1$

$x^2+y^2+z^2=4 \longrightarrow \rho^2=4$
$\rho=2$

$\boxed{1\le\rho\le 2}$

$\boxed{0\le\theta\le\frac{\pi}{2}}$

$\boxed{0\le\varphi\le\frac{\pi}{2}}$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^{2} \underbrace{\rho\cos\theta\sin\varphi}_{x}\, e^{(\rho^2)^2}\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\int_1^{2}\rho^3 e^{\rho^4}\cos\theta\,\sin^2\varphi\,d\rho\,d\varphi\,d\theta$$

$$=\int_0^{\pi/2}\cos\theta\,d\theta\int_0^{\pi/2}\sin^2\varphi\,d\varphi\int_1^{2}\rho^3 e^{\rho^4}\,d\rho$$

$u=\rho^4,\quad du=4\rho^3 d\rho \longrightarrow \rho^3 d\rho=\frac{du}{4}$

$\rho=1\longrightarrow u=1^4=1$

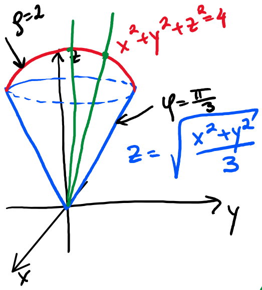
$\rho=2\longrightarrow u=2^4=16$

$$=\sin\theta\Big|_0^{\pi/2}\int_0^{\pi/2}\frac{1-\cos 2\varphi}{2}\,d\varphi\int_1^{16}e^u\frac{du}{4}$$

$$=\left(\sin\frac{\pi}{2}-\sin 0\right)\frac{1}{2}\left(\varphi-\frac{1}{2}\sin 2\varphi\right)\Big|_0^{\pi/2}\frac{e^u}{4}\Big|_1^{16}$$

$$=\frac{1}{2}\left(\frac{\pi}{2}\right)\frac{1}{4}\left(e^{16}-e\right)=\boxed{\frac{\pi}{16}\left(e^{16}-e\right)}$$

8. Use cylindrical and spherical coordinates to find the volume of the ice-cream cone bounded above by the sphere $x^2+y^2+z^2=4$ and below by the cone $z=\sqrt{\dfrac{x^2+y^2}{3}}$.



Spherical

$$\left[\begin{aligned} x &= \rho\cos\theta\sin\varphi \\ y &= \rho\sin\theta\sin\varphi \\ z &= \rho\cos\varphi \\ dV &= \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta \end{aligned}\right]$$

$$x^2+y^2+z^2=4 \longrightarrow \rho=2$$

$$\rho\cos\varphi = \sqrt{\frac{\rho^2\cos^2\theta\sin^2\varphi+\rho^2\sin^2\theta\sin^2\varphi}{3}}$$

$$= \sqrt{\frac{\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)}{3}}$$

$$\frac{\sqrt{3}}{\cancel{\cos\varphi}}\,\cancel{\rho\cos\varphi} = \frac{\cancel{\rho}\sin\varphi}{\cancel{\sqrt{3}}}\,\frac{\cancel{\sqrt{3}}}{\cos\varphi}$$

$$\tan\varphi = \sqrt{3} \Rightarrow \varphi = \frac{\pi}{3}$$

[origin] $\le \rho \le$ [the sphere]

$$\boxed{0\le\rho\le 2}$$

$$0 \le \varphi \le \frac{\pi}{3}$$

[positive z-axis] $\le \varphi \le$ [the cone]

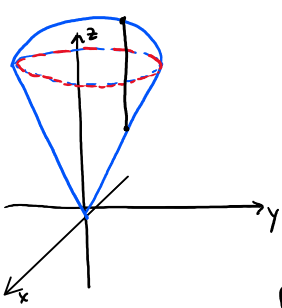
$$0\le\theta\le 2\pi$$

$$V = \iiint_E 1\cdot dV = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{2} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/3}\sin\varphi\, d\varphi \int_0^2 \rho^2\, d\rho$$

$$= 2\pi\left(-\cos\varphi\right)\Big|_0^{\pi/3}\ \frac{\rho^3}{3}\Big|_0^2$$

$$= 2\pi\left(-\cos\frac{\pi}{3}+\cos 0\right)\frac{8}{3} = 2\pi\left(\frac{1}{2}\right)\frac{8}{3}$$



cylindrical coordinates.

$$\left[\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ dV &= r\, dz\, dr\, d\theta \end{aligned}\right]$$

$$x^2+y^2+z^2=4$$

$$r^2+z^2=4$$

$$z^2=4-r^2$$

$$z=\sqrt{4-r^2}$$

$$z=\sqrt{\frac{x^2+y^2}{3}}$$

$$z=\sqrt{\frac{r^2}{3}}$$

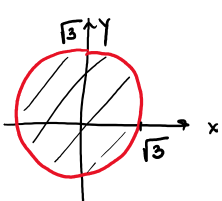
$$z=\frac{r}{\sqrt{3}}$$

$$r^2+\frac{r^2}{3}=4$$

$$\frac{4r^2}{3}=4 \Rightarrow r^2=3$$

[cone] $\le z \le$ [sphere]

$$\boxed{\frac{r}{\sqrt{3}} \le z \le \sqrt{4-r^2}}$$



$$\left[\begin{aligned} 0 &\le r \le \sqrt{3} \\ 0 &\le \theta \le 2\pi \end{aligned}\right]$$

$$V = \iiint_E dV = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{r/\sqrt{3}}^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} rz\Big|_{r/\sqrt{3}}^{\sqrt{4-r^2}} dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}}\left(r\sqrt{4-r^2}-\frac{r^2}{\sqrt{3}}\right)dr\, d\theta$$

$[0 \le \theta \le 2\pi]$

$$= \int_0 \int_0 r\, z \Big|_{\frac{r}{\sqrt{3}}} \, dr\, d\theta = \int_0 \int_0 \left(r\sqrt{4-r^2} - \frac{r^2}{\sqrt{3}} \right) dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \left[\int_0^{\sqrt{3}} r\sqrt{4-r^2}\, dr - \frac{1}{\sqrt{3}} \int_0^{\sqrt{3}} r^2\, dr \right]$$

$$\left[\begin{array}{l} u = 4 - r^2 \\ du = -2r\,dr \\ r\,dr = -\frac{du}{2} \\ r = 0 \rightarrow u = 4 - 0 = 4 \\ r = \sqrt{3} \rightarrow u = 4 - 3 = 1 \end{array} \right]$$

$$= 2\pi \left[\int_4^1 \sqrt{u} \left(-\frac{du}{2} \right) - \frac{1}{\sqrt{3}} \frac{r^3}{3} \Big|_0^{\sqrt{3}} \right]$$

$$= 2\pi \left[\frac{1}{2} \int_1^4 \sqrt{u}\, du - 1 \right]$$

$$= 2\pi \left(\frac{1}{2} \frac{u^{3/2}}{3/2} \Big|_1^4 - 1 \right)$$

$$= 2\pi \left(\frac{1}{3} \left(4^{3/2} - 1 \right) - 1 \right)$$

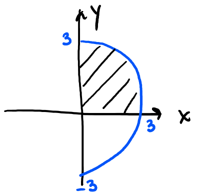
$$= 2\pi \left(\frac{7}{3} - 1 \right) = \frac{8\pi}{3}$$

7. Convert the integral

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{\sqrt{9-x^2-y^2}} \underbrace{(x^2+y^2+z^2)}_{\rho^2}\, \underbrace{dz\,dx\,dy}_{\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta}$$

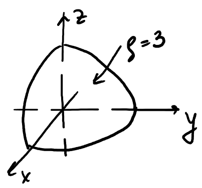
to an integral in spherical coordinates, but don't evaluate it.

$$0 \le z \le \underbrace{+\sqrt{9-x^2-y^2}}_{\text{top hemisphere}} \Rightarrow \underbrace{z^2+x^2+y^2}_{\substack{\rho^2=9\\ \rho=3}}=9 \longrightarrow \left[\begin{array}{l} x=\rho\cos\theta\sin\varphi \\ y=\rho\sin\theta\sin\varphi \\ z=\rho\cos\varphi \\ dV=\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta \end{array}\right]$$



$$0 \le x \le +\sqrt{9-y^2} \longrightarrow x^2+y^2=9$$

$$0 \le y \le 3$$

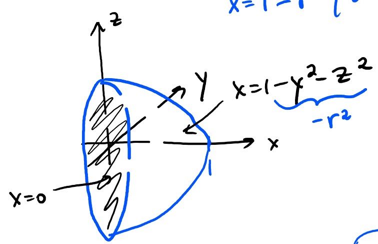


$$\left[\begin{array}{l} 0 \le \rho \le 3 \\ 0 \le \theta \le \pi/2 \\ 0 \le \varphi \le \pi/2 \end{array}\right] \qquad \boxed{= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho^4 \sin\varphi \, d\rho\, d\varphi\, d\theta}$$

6. Evaluate $\iiint_E y^2 z^2\, dV$, where E is the solid bounded by the paraboloid $x = 1 - y^2 - z^2$, and the plane $x = 0$.

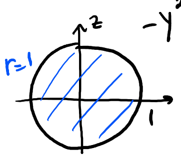
$x = 1 - r^2$ (the paraboloid)

cylindrical coordinates



$$\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \end{cases}$$

$dV = r\,dx\,dr\,d\theta$



$-y^2 - z^2 + 1 = 0$

$\underbrace{y^2 + z^2}_{} = 1$

$r^2 = 1$

$$\begin{bmatrix} 0 \le x \le 1 - r^2 \\ 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{bmatrix}$$

$$\int_0^{2\pi}\int_0^1\int_0^{1-r^2} (\underbrace{r^2\cos^2\theta}_{y^2})(\underbrace{r^2\sin^2\theta}_{z^2})\, r\, dx\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^5 \cos^2\theta \sin^2\theta\, (x)\Big|_0^{1-r^2} dr\,d\theta$$

$$= \int_0^{2\pi} \underbrace{\cos^2\theta \sin^2\theta}_{}\, d\theta \int_0^1 r^5(1-r^2)\, dr$$

$\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$

$\cos^2\theta\sin^2\theta = \frac{1}{4}\sin^2 2\theta$

$$= \int_0^{2\pi} \frac{1}{4}\sin^2 2\theta\, d\theta \int_0^1 (r^5 - r^7)\, dr$$

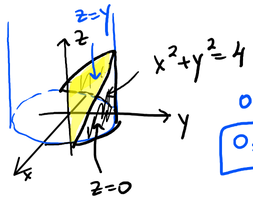
$\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$

$$= \frac{1}{8}\int_0^{2\pi} (1 - \cos 4\theta)\, d\theta \left(\frac{r^6}{6} - \frac{r^8}{8}\right)\Big|_0^1$$

$$= \frac{1}{8}\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_0^{2\pi} \left(\frac{1}{6} - \frac{1}{8}\right)$$

$$= \frac{2\pi}{8}\left(\frac{1}{6} - \frac{1}{8}\right)$$

5. Evaluate $\iiint_E yz\,dV$, where E lies above the plane $\underbrace{z=0}_{(xy)\text{-plane}}$, below the plane $z=y$ and inside the cylinder $x^2+y^2=4$.



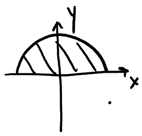
$0 \le z \le y$

$\boxed{0 \le z \le r\sin\theta}$

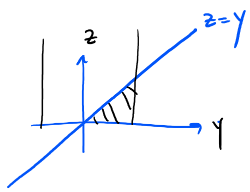
$$\begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \\ z = z \\ dV = r\,dz\,dr\,d\theta \end{bmatrix}$$

$x^2+y^2=4 \longrightarrow r^2=4$

$r=2$



$$\begin{bmatrix} 0 \le r \le 2 \\ 0 \le \theta \le \pi \end{bmatrix}$$



$$= \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta} z\,(r\sin\theta)\, r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{2} r^2\sin\theta\, \frac{z^2}{2}\Big|_0^{r\sin\theta} dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{2} r^2\sin\theta\,\frac{1}{2}\, r^2\sin^2\theta\, dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta\sin^2\theta\,d\theta \int_0^{2} r^4\,dr$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta(1-\cos^2\theta)\,d\theta \cdot \frac{r^5}{5}\Big|_0^{2}$$

$u=\cos\theta$

$du = -\sin\theta\,d\theta$

$\theta = 0 \rightarrow u=\cos 0 = 1$

$\theta = \pi \rightarrow u = \cos\pi = -1$

$$= \frac{1}{2}\int_{+1}^{-1} (1-u^2)(-du)\cdot\left(\frac{32}{5}\right)$$

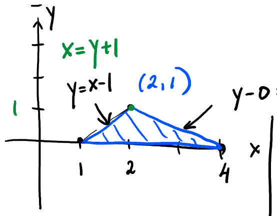
$$= \frac{1}{2}\int_{-1}^{1} (1-u^2)\,du \left(\frac{32}{5}\right)$$

$$= \frac{1}{2}\left(u-\frac{u^3}{3}\right)\Big|_{-1}^{1} \left(\frac{32}{5}\right)$$

$$= \frac{16}{5}\left(2-\frac{2}{3}\right) = \frac{64}{15}$$

4. Find the volume of the solid under $z = x^2y$ and above the triangle in the (xy)-plane with vertices (1,0), (2,1), (4,0).

$0 \le z \le x^2 y$



$y - 0 = \frac{1-0}{2-4}(x-4)$

$y = -\frac{1}{2}(x-4)$

$y = -\frac{1}{2}x + 2 \Rightarrow y - 2 = -\frac{1}{2}x$

$x = 4 - 2y$

$y+1 \le x \le 4-2y$

$0 \le y \le 1$

$$V = \int_0^1 \int_{y+1}^{4-2y} \int_0^{x^2y} 1 \, dz \, dx \, dy$$

$$= \int_0^1 \int_{y+1}^{4-2y} x^2 y \, dx \, dy$$

$$= \int_0^1 \frac{x^3}{3} \Big|_{y+1}^{4-2y} y \, dy$$

$$= \frac{1}{3} \int_0^1 \left[(4-2y)^3 - (y+1)^3 \right] y \, dy$$

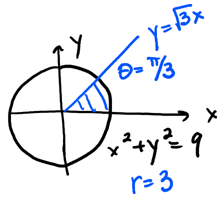
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$= \frac{1}{3} \int_0^1 \left[64 - 3(16)(2y) + 4(4y^2) - 8y^3 - (y^3 + 3y^2 + 3y + 1) \right] y \, dy$$

$$= \frac{1}{3} \int_0^1 \left[63 - 99y + 13y^2 - 9y^3 \right] y \, dy = \ldots$$

3. Evaluate the integral $\iint_D (x^2+y^2)^{3/2}dA$, where D is the region bounded by the lines $\underbrace{y=0}_{x\text{-axis}}$, $y=\sqrt{3}x$, and the circle $x^2+y^2=9$.



polar coordinates

$$\begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \\ dA = r\,dr\,d\theta \end{bmatrix}$$

$0 \le r \le 3$

$0 \le \theta \le \frac{\pi}{3}$

$y = \sqrt{3}x$

$$\frac{\cancel{r}\sin\theta}{\cos\theta} = \frac{\sqrt{3}\,\cancel{r}\cos\theta}{\cos\theta}$$

$\tan\theta = \sqrt{3}$

$\boxed{\theta = \pi/3}$

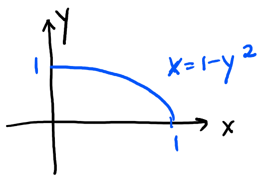
$$\iint_D (x^2+y^2)^{3/2}\,dA = \int_0^{\pi/3}\int_0^3 r^3\,r\,dr\,d\theta$$

$$= \int_0^{\pi/3} d\theta \int_0^3 r^4\,dr$$

$$= \theta\Big|_0^{\pi/3}\ \frac{r^5}{5}\Big|_0^3$$

$$= \frac{\pi}{3}\cdot\frac{81\cdot 3}{5} = \boxed{\frac{81\pi}{5}}$$

2. Evaluate the integral $\iint_D (xy + 2x + 3y)\,dA$, where D is the region bounded by $x = 1 - y^2$, $\underbrace{y = 0}_{\text{x-axis}}$, $\overbrace{x = 0}^{\text{y-axis}}$.



$0 \le x \le 1-y^2$

$0 \le y \le 1$

$$\int_0^1 \int_0^{1-y^2} (xy + 2x + 3y)\,dx\,dy = \ldots$$

Review for Exam 3.

1. Calculate the iterated integral $\int_0^1 \int_x^1 e^{x/y}\, dy\, dx$ by reversing the order of integration.

$0 \le x \le 1$, $x \le y \le 1$ | $0 \le x \le y$, $0 \le y \le 1$

$$= \int_0^1 \int_0^y e^{x/y}\, dx\, dy$$

$$= \int_0^1 y e^{x/y} \Big|_0^y dy$$

$$= \int_0^1 y\left(e^{y/y} - e^0\right) dy$$

$$= \int_0^1 y(e-1)\, dy$$

$$= \boxed{\frac{1}{2}(e-1)}$$