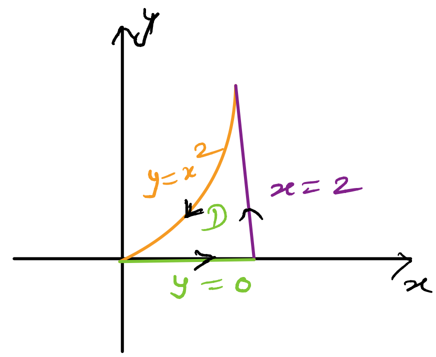
Example 1 (16.4)**.** *Use the Green's Theorem to compute the line integral* $\oint_C \mathbf{F} \cdot d\mathbf{r}$*, where* $\mathbf{F}(x, y) = (x^2y^2 + x^2 \sin x)\,\mathbf{i} + (2x^3y + e^y)\,\mathbf{j}$ *and* C *is the boundary of the region bounded by the curves* $y = x^2,\ x = 2,$ *and* $y = 0$.



$$P = x^2y^2 + x^2 \sin x \Rightarrow P_y = 2x^2y$$

$$Q = 2x^3y + e^y \Longrightarrow Q_x = 6x^2y$$

$$\oint_C F \cdot dr \overset{\text{Green's}}{=} \iint_D (Q_x - P_y)\, dA$$

$$= \iint_D 4x^2y\, dA = \int_0^2 \int_0^{x^2} 4x^2y\, dy\, dx$$

$$= 2\int_0^2 x^2 \left[y^2\right]_0^{x^2} dx = 2\int_0^2 x^6\, dx$$

$$= 2\left.\frac{x^7}{7}\right|_0^2 = \frac{256}{7}$$

Example 2 (16.4)**.** *Use the Green's Theorem to compute the line integral* $\oint_C \mathbf{F} \cdot d\mathbf{r}$*, where* $\mathbf{F}(x, y) = \left\langle e^{3x} - 4y, 6x + e^{2y} \right\rangle$ *and* C *is the boundary of the region that has area 4 with counterclockwise orientation.*

$$P = e^{3x} - 4y \Rightarrow P_y = -4$$

$$Q = 6x + e^{2y} \Rightarrow Q_x = 6$$

$$\oint_C F \cdot dr \overset{\text{Green's}}{=} \iint_D [6 - (-4)]\, dA,$$ where D is the region bounded by C.

$$= 10 \iint_D dA = 10\, A(D) = 10 \cdot 4 = 40$$

Example 3 (16.4)**.** *Use the Green's Theorem to compute* $\int_C \underbrace{(3xy^2-2y^3)}_{P}\,dx+\underbrace{(2x^3+3x^2y)}_{Q}\,dy$, *where C is the circle $x^2+y^2=9$ with positive orientation.*

$$\oint_C (3x^2y-2y^3)\,dx+(2x^3+3x^2y)\,dy \overset{\text{Green's}}{=} \iint_D \left[6x^2+\cancel{6xy}-(\cancel{6xy}-6y^2)\right] dA$$

where $D: x^2+y^2\le 9$

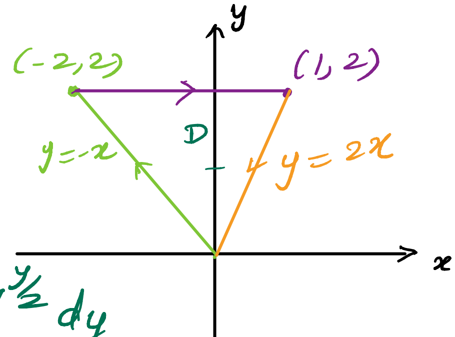
$$= 6\iint (x^2+y^2)\,dA \qquad \text{i.e. } 0\le r\le 3,\ 0\le\theta\le 2\pi$$

$$= 6\int_0^{2\pi}\int_0^3 r^2\cdot r\,dr\,d\theta$$

$$= 6\left(\int_0^{2\pi} d\theta\right)\left(\int_0^3 r^3\,dr\right) = 6\cdot 2\pi\cdot \frac{r^4}{4}\Big|_0^3$$

$$= 243\pi$$

Example 4 (16.4)**.** *Compute* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F}(x,y)=\langle \overbrace{2xy^2}^{P}, \overbrace{3x^2y-5}^{Q}\rangle$ *and C is the triangle from $(0,0)$ to $(-2,2)$ to $(1,2)$ to $(0,0)$.*



$$\int_C F\cdot dr = -\int_{-C} F\,dr \overset{\text{Green's}}{=} -\iint_D (6xy-4xy)\,dA$$

has +ve orientation

$$= -\int_0^2\int_{-y}^{y/2} 2xy\,dx\,dy = -\int_0^2 y\left[x^2\right]_{-y}^{y/2} dy$$

$$= -\int_0^2 y\left[\frac{y^2}{4}-y^2\right] dy = \frac{3}{4}\int_0^2 y^3\,dy = \frac{3}{4}\,\frac{y^4}{4}\Big|_0^2$$

$$= 3$$

Example 5 (16.5)**.** *Find the curl and divergence of*

$$\mathbf{F}(x,y,z) = \underbrace{xyz}_{P}\,\mathbf{i} + \underbrace{(x^2+yz)}_{Q}\,\mathbf{j} + \underbrace{xz}_{R}\,\mathbf{k}.$$

Is $\mathbf{F}$ *conservative? Explain.*

$\operatorname{div} F = P_x + Q_y + R_z = yz + z + x$, a scalar function.

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ xyz & x^2+yz & xz \end{vmatrix} = \langle -y, -(z-xy), 2x-xz \rangle$$

Since F is defined everywhere on $\mathbb{R}^3$ (which is simply connected) and $\operatorname{curl} F \neq \vec{0}$, F is not conservative.

Example 6 (16.5)**.** *Let* f *be a scalar function and* $\mathbf{F}$ *and* $\mathbf{G}$ *are vector fields on* $\mathbb{R}^3$*. State whether each expression is meaningful. If so, state whether it;s a vector field or a scalar field.*

(a) $\underbrace{\nabla f}_{\text{vector}} \times \underbrace{(\mathbf{F}+\mathbf{G})}_{\text{vector}}$ $\rightsquigarrow$ Yes, a vector field.

(b) $\underbrace{\nabla f}_{\text{Vector}} \cdot \underbrace{curl\,\mathbf{F}}_{\text{vector}}$ $\rightsquigarrow$ Yes, a scalar function.

(c) $\underbrace{\nabla f}_{\text{vector}} \times \underbrace{div\,\mathbf{F}}_{\text{scalar}}$ $\rightsquigarrow$ Not meaningful.

(d) $(\underbrace{curl\,\mathbf{F} \times \mathbf{G}}_{\text{vector}}) \cdot \underbrace{\nabla f}_{\text{vector}}$ $\rightsquigarrow$ Yes, a scalar function.

(e) $curl\,(\underbrace{\mathbf{F} \cdot \mathbf{G}}_{\text{scalar}})$ $\rightsquigarrow$ Not meaningful.

(f) $div\,(curl\,F \times \nabla f)$ $\rightsquigarrow$ Yes, a scalar function.

Example 7 (16.5)**.** *Consider the vector field* $\mathbf{F}(x,y,z) = \langle 2xy+3, x^2+z\cos y, \sin y\rangle$.

(a) Determine whether or not $\mathbf{F}$ *is conservative. If it is, find a potential function* f*. That is, find a function* f *such that* $\nabla f = \mathbf{F}$.

$$\text{Curl } F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xy+3 & x^2+z\cos y & \sin y \end{vmatrix} = 0i + 0j + 0k = \vec{0}.$$

Since F is defined everywhere on $\mathbb{R}^3$ and curl $F = 0$, it is conservative.

We want $f(x,y,z)$ such that $\nabla f = F$. i.e.

$$f_x = 2xy + 3 \quad \text{— (1)}$$
$$f_y = x^2 + z\cos y \quad \text{— (2)}$$
$$f_z = \sin y \quad \text{— (3)}$$

Let's integrate (2) w.r.t y; $f(x,y,z) = \int x^2 + z\cos y \, dy$

$$\Rightarrow f(x,y,z) = x^2y + z\sin y + g(x,z) \quad \text{— (4)}$$

$$2x\cancel{y} + 3 \overset{(1)}{=} f_x \overset{(4)}{=} \cancel{2xy} + 0 + g_x(x,z) \Rightarrow g_x(x,z) = 3$$

$$\Rightarrow \int g_x(x,z)\,dx = \int 3\,dx = 3x + h(z)$$ Trial and error $h(z) = 0$ works here.

So, $f(x,y,z) = x^2y + z\sin y + 3x$ is a potential function of F.

(b) Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $C:\ r(t) = \langle t,\ 2t,\ 1+t^2\rangle;\ 0 \le t \le \pi$.

$r(0) = (0,0,1)$ ⇝ initial point.

$r(\pi) = (\pi, 2\pi, 1+\pi^2)$ ⇝ terminal point.

$$\int_C F\cdot dr = \int_C \nabla f \cdot dr \overset{FTLI}{=} f(\text{terminal point}) - f(\text{initial point})$$
$$= f(\pi, 2\pi, 1+\pi^2) - f(0,0,1)$$
$$= (\pi^2\cdot(2\pi) + 0 + 3\pi) - 0$$
$$= 2\pi^3 + 3\pi$$

Example 8 (16.6). *Find a parametric representation for the surface.*

(a) *The part of the plane $2x + z = 8$ that lies within the cylinder $x^2 + y^2 = 9$.*

(b) *The part of the cylinder $x^2 + y^2 = 9$ within the planes $z = 0$ and $z = 3$.*

(c) *The part of the cylinder $y^2 + z^2 = 9$ within the planes $x = 0$ and $x = 3$.*

(d) *The part of the paraboloid $z = 6 - 2x^2 - 2y^2$ above the plane $z = 4$.*

(a) $2x + z = 8 \Rightarrow z = f(x,y) = 8 - 2x$

$r(x,y) = xi + yj + f(x,y)k = xi + yj + (8-2x)k$

over $D: x^2 + y^2 \leq 9$.

Further, if we use polar coordinates with

$x = u\cos v,\ y = u\sin v, \quad 0 \leq u \leq 3,\ 0 \leq v \leq 2\pi$

$\Rightarrow r(u,v) = \langle u\cos v,\ u\sin v,\ 8 - 2u\cos v \rangle$ " "

(b) $x^2 + y^2 = 9 \Rightarrow x = 3\cos u,\ y = 3\sin u,\ z = v$

$r(u,v) = \langle 3\cos u,\ 3\sin u,\ v \rangle;\ 0 \leq u \leq 2\pi,\ 0 \leq v \leq 3$.

(c) $y^2 + z^2 = 9 \Rightarrow y = 3\cos v,\ z = 3\sin v, \quad 0 \leq v \leq 2\pi$
$x = u, \quad 0 \leq u \leq 3$.

$r(u,v) = \langle u,\ 3\cos v,\ 3\sin v \rangle;\ 0 \leq u \leq 3,\ 0 \leq v \leq 2\pi$.

(d) The intersection of $z = 4$ and $z = 6 - 2x^2 - 2y^2$ is $x^2 + y^2 = 1$.
The projection of the surface onto xy-plane is $x^2 + y^2 \leq 1$.

$r(x,y) = xi + yj + (6 - 2x^2 - 2y^2)k;\ D: x^2 + y^2 \leq 1$.

Using polar coordinates: $x = u\cos v,\ y = u\sin v, \quad 0 \leq u \leq 1,\ 0 \leq v \leq 2\pi$

$r(u,v) = \langle u\cos v,\ u\sin v,\ 6 - 2u^2 \rangle;\ 0 \leq u \leq 1,\ 0 \leq v \leq 2\pi$.

Example 9 (16.6). *Find the surface area of the part of the plane $2x + 3y + z = 8$ that lies within the cylinder $x^2 + y^2 = 4$.*

$2x + 3y + z = 8 \Rightarrow z = g(x,y) = 8 - 2x - 3y$

$r(x,y) = \langle x, y, 8 - 2x - 3y \rangle$; $D = \{(x,y) : x^2 + y^2 \leq 4\}$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{vmatrix} = \langle 2, 3, 1 \rangle .$$

$$|r_x \times r_y| = \sqrt{1 + 4 + 9} = \sqrt{14}$$

In fact, if the surface is given by $z = g(x,y)$, then

$r_x \times r_y = \langle -g_x, -g_y, 1 \rangle$ and

$|r_x \times r_y| = \sqrt{1 + g_x^2 + g_y^2}$.

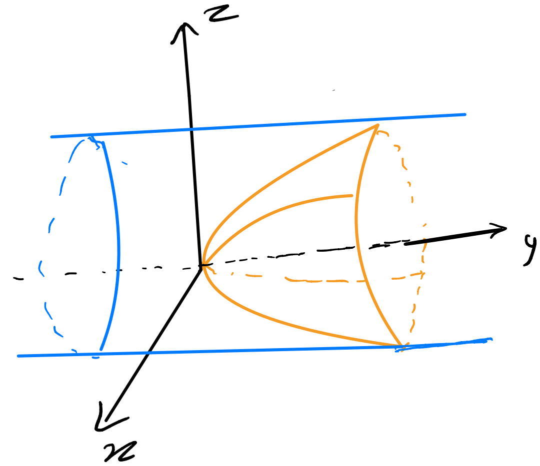
$$A(S) = \iint_D |r_x \times r_y| \, dA = \iint_D \sqrt{14} \, dA$$

$$= \sqrt{14} \iint_D dA = \sqrt{14} \cdot A(D)$$

$$= \sqrt{14} \cdot (\pi \cdot 2^2)$$

$$= 4\sqrt{14}\,\pi$$

Example 10 (16.6). *Find the surface area of S, where S is the part of the paraboloid $y = x^2 + z^2$ that lies within the cylinder $x^2 + z^2 = 4$.*



$$r(x,z) = x\,i + (x^2+z^2)\,j + z\,k;$$

$$D: x^2+z^2 \le 4$$

$$r_x \times r_z = \begin{vmatrix} i & j & k \\ 1 & 2x & 0 \\ 0 & 2z & 1 \end{vmatrix}$$

$$= \langle 2x, -1, 2z \rangle$$

$$|r_x \times r_z| = \sqrt{1+4x^2+4z^2}$$

In fact:

$$r_x \times r_z = \sqrt{1+y_x^2+y_z^2}$$

$$= \sqrt{1+4x^2+4z^2}.$$

$$A(S) = \iint_D |r_x \times r_z|\,dA = \iint_{x^2+z^2\le 4} \sqrt{1+4x^2+4z^2}\,dA$$

Using polar coordinates: $x = r\cos\theta$, $z = r\sin\theta$,

$0 \le r \le 2$, $0 \le \theta \le 2\pi$

$$A(S) = \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\cdot r\,dr\,d\theta$$

$u = 1+4r^2 \Rightarrow du = 8r\,dr$

$$= 2\pi\cdot\frac{1}{8}\int \sqrt{u}\,du = \frac{\pi}{4}\cdot\frac{2}{3}\left[(1+4r^2)^{\frac{3}{2}}\right]_0^2$$

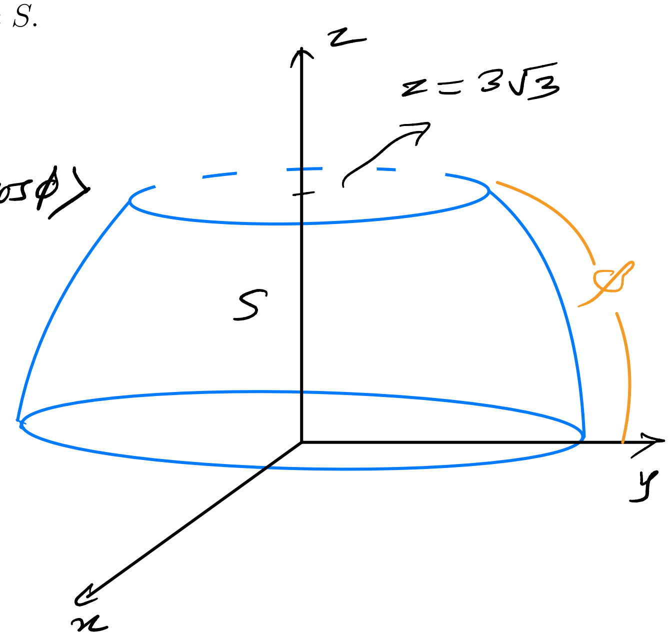
$$= \frac{\pi}{6}\left[(17)^{\frac{3}{2}} - 1\right]$$

$\rho = 6$

Example 11 (16.6)**.** *Consider that S is the part of the sphere $x^2 + y^2 + z^2 = 36$ that lies within the planes $z = 0$ and $z = 3\sqrt{3}$.*

(a) Find a parametric representation for the surface S.

(b) Find the surface area of the surface S.



(a)

$r(\phi, \theta) = \langle 6\sin\phi\cos\theta, 6\sin\phi\sin\theta, 6\cos\phi \rangle$

Clearly, $0 \le \theta \le 2\pi$.

$z = 3\sqrt{3} \Rightarrow 6\cos\phi = 3\sqrt{3}$

$\Rightarrow \cos\phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}$

$z = 0 \Rightarrow \phi = \frac{\pi}{2}$

So, $\frac{\pi}{6} \le \phi \le \frac{\pi}{2}$

That is, $D = \{(\phi, \theta) : \frac{\pi}{6} \le \phi \le \frac{\pi}{2}, 0 \le \theta \le 2\pi\}$.

(b) Recall that $|r_\phi \times r_\theta| = \rho^2 \sin\phi = 36\sin\phi$.

$$A(S) = \iint_D |r_\phi \times r_\theta|\, dA = 36\int_0^{2\pi}\int_{\pi/6}^{\pi/2} \sin\phi \, d\phi\, d\theta$$

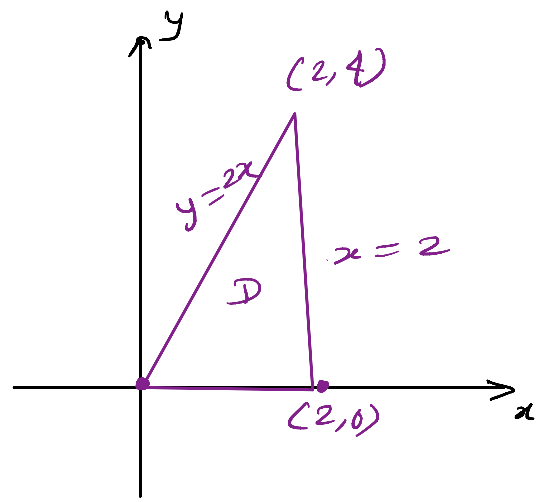
$$= 36\left(\int_0^{2\pi} d\theta\right)\left(\int_{\pi/6}^{\pi/2} \sin\phi\, d\phi\right)$$

$$= 36 \cdot 2\pi \cdot -\cos\phi \Big|_{\pi/6}^{\pi/2} = 72\pi\left[0 + \frac{\sqrt{3}}{2}\right]$$

$$= 36\sqrt{3}\,\pi$$

Example 12 (16.6)**.** *Find the area of the part of the surface $z = 1 + 2x^2 + 3y$ that lies above the region bounded the triangle with vertices* $(0, 0)$, $(2, 0)$, *and* $(2, 4)$.

$$r(x,y) = \langle x,\ y,\ 1+2x^2+3y \rangle$$



$$|r_x \times r_y| = \sqrt{1 + z_x^2 + z_y^2}$$

$$= \sqrt{1 + 16x^2 + 9}$$

$$= \sqrt{10 + 16x^2}$$

$$A(S) = \iint_D |r_x \times r_y|\, dA = \iint_D \sqrt{10+16x^2}\, dA$$

$$= \int_0^2 \int_0^{2x} \sqrt{10+16x^2} \cdot dy\, dx$$

$$= \int_0^2 2x\sqrt{10+16x^2}\, dx$$

$u = 10 + 16x^2$

$du = 32x\, dx$

$$= \frac{1}{16}\int \sqrt{u}\, du$$

$$= \frac{1}{16} \cdot \frac{2}{3} \left[(10+16x^2)^{3/2} \right]_0^2$$

$$= \frac{1}{24} \left[(74)^{3/2} - 10^{3/2} \right]$$