$\sum_{n=0}^{\infty} c_n (x-a)^n$

convergent if $|x-a| < R$, R is the radius of convergence.

$-R < x-a < R$

$-R+a < x < R+a$ interval of convergence.

Test the points $x = a \pm R$ separately.

Find the radius of convergence either using the Ratio Test or using the Root Test for $a_n = c_n (x-a)^n$

1. Determine the radius and interval of convergence for the series.

(a) $\sum_{n=0}^{\infty} \frac{(x+5)^n}{n!}$

$a_n = \frac{(x+5)^n}{n!}$, $a_{n+1} = \frac{(x+5)^{n+1}}{(n+1)!}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x+5)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x+5)^n}\right|$ $\underline{\underline{(n+1)! = (n+1)n!}}$ $\lim_{n\to\infty} \left|\frac{x+5}{n+1}\right| = \boxed{0<1}$ for all x.

The series converges for all x.

$$\boxed{\begin{array}{ll} \text{Interval of convergence} & (-\infty, \infty) \\ \text{Radius of convergence} & R = \infty \end{array}}$$

(b) $\sum_{n=0}^{\infty} n!(2x-1)^n$

$a_n = n!(2x-1)^n$, $a_{n+1} = (n+1)!\,(2x-1)^{n+1}$

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\,(2x-1)^{\cancel{n+1}}}{n!\,\cancel{(2x-1)^n}}\right|$ $\underline{\underline{(n+1)! = (n+1)n!}}$

$= \lim_{n\to\infty} \left|\frac{(n+1)n!\,(2x-1)}{n!}\right| = \lim_{n\to\infty} \left|(n+1)(2x-1)\right| = \infty > 0$

The series diverges everywhere except $2x-1=0$.

$x = \frac{1}{2}$

$$\boxed{\begin{array}{ll} \text{Interval of convergence is} & \{\tfrac{1}{2}\} \\ \text{Radius of convergence} & R = 0 \end{array}}$$

(c) $\sum_{n=0}^{\infty} \frac{n(x-2)^n}{2^n(n^2+1)}$

$a_n = \frac{n(x-2)^n}{2^n(n^2+1)}$, $a_{n+1} = \frac{(n+1)(x-2)^{n+1}}{2^{n+1}[(n+1)^2+1]}$

Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-2)^{n+1}}{2^{n+1}[(n+1)^2+1]}\cdot\frac{2^n(n^2+1)}{n(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(n^2+1)(x-2)}{2[(n+1)^2+1]\,n}\right|$$

$$= \frac{|x-2|}{2}\lim_{n\to\infty}\left|\cancel{\frac{(n+1)(n^2+1)}{n(n^2+2n+2)}}^{\,1}\right| = \frac{|x-2|}{2} < 1$$

$|x-2| < 2 \leftarrow R=2$ radius of convergence.

$-2 < x-2 < 2$

$\boxed{0 < x < 4}$ interval of convergence.

Test the end-points of the interval of convergence.

$\boxed{x=0}$, $\sum_{n=0}^{\infty}\frac{n(0-2)^n}{2^n(n^2+1)} = \sum_{n=0}^{\infty}\frac{(-2)^n\cdot n}{2^n(n^2+1)}$ $\quad \underline{(-2)^n = (-1\cdot 2)^n = (-1)^n\cdot 2^n}$ $\quad \sum_{n=0}^{\infty}\frac{(-1)^n \cancel{2^n} n}{\cancel{2^n}(n^2+1)}$

$= \sum_{n=0}^{\infty}\frac{(-1)^n n}{n^2+1}$ alternating series.

Alternating series test.

$\sum_{n=1}(-1)^{n-1} b_n$

$\left[\begin{array}{l}(1)\ \lim_{n\to\infty} b_n = 0 \\ (2)\ b_{n+1} < b_n\end{array}\right] \longrightarrow \sum_{n=1}^{\infty}(-1)^{n-1} b_n$ is convergent

(1) $\lim_{n\to\infty}\left|\frac{(-1)^n n}{n^2+1}\right| = \lim_{n\to\infty}\left|\frac{n}{n^2+1}\right| = 0$

(2) $b_n = \frac{n}{n^2+1}$, $b_{n+1} = \frac{n+1}{(n+1)^2+1}$

differentiate for n $\rightarrow (b_n)' = \left(\frac{n}{n^2+1}\right)' = \frac{(n^2+1)-2n\cdot n}{(n^2+1)^2} = \frac{n^2+1-2n^2}{(n^2+1)^2} = \frac{1-n^2}{(n^2+1)^2} < 0$ whenever

$1-n^2<0$

$n^2>1$

$(n=2,3,4,5,\ldots)$

Starting with $n=2$ $\quad b_{n+1} < b_n$.

$\boxed{\sum_{n=0}\frac{(-1)^n n}{n^2+1} \text{ converges by the Alternating Series Test.}}$

$\boxed{x=4}$ $\sum_{n=0}\frac{n(4-2)^n}{2^n(n^2+1)} = \sum_{n=0}\frac{n\cdot\cancel{2^n}}{\cancel{2^n}(n^2+1)} = \sum_{n=0}^{\infty}\frac{n}{n^2+1}$ compare with $\sum_{n=1}^{\infty}\frac{n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n}$ – harmonic series divergent.

Limit Comparison Test.

$\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$

If $\lim_{n\to\infty}\frac{a_n}{b_n} = c > 0$, and $c \neq \infty$, then these series either both converge or they both diverge

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$a_n = \frac{n}{n^2+1}$, $b_n = \frac{1}{n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1>0,\ 1\neq\infty.$$

$\sum_{n=0}\frac{n}{n^2+1}$ is divergent by the Limit Comparison Test.

Update the interval of convergence: $\boxed{0 \le x < 4}$

Radius of convergence $\boxed{R=2}$

Update the interval of convergence: $\boxed{0 \le x < 4}$

Radius of convergence $\boxed{R=2}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

$$\boxed{\frac{1}{1-\text{☺}} = \sum_{n=0}^{\infty} (\text{☺})^n}, \quad |\text{☺}|<1$$

$$\boxed{\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n}$$

Inside the interval of convergence, we can differentiate and integrate a power series term by term.

$$\left(\frac{1}{1-x}\right)' = -\frac{1}{(1-x)^2}$$

$|x|<1$ $$\left(\frac{1}{1-x}\right)' = \left(\sum_{n=0}^{\infty} x^n\right)' \longrightarrow -\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\boxed{\frac{1}{(1-x)^2} = -\sum_{n=1}^{\infty} n x^{n-1}}$$

$$\left(\frac{x^2}{25}\right)^n = \frac{x^{2n}}{25^n} = \frac{x^{2n}}{5^{2n}} = \left(\frac{x}{5}\right)^{2n}$$

2. Find a power series representation for the function and determine the interval of convergence.

(a) $\dfrac{x}{x^2+25}$ $= \dfrac{x}{25+x^2} = \dfrac{x}{25\left(1+\frac{x^2}{25}\right)} = \dfrac{x}{25}\dfrac{1}{1+\frac{x^2}{25}} = \dfrac{x}{25}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{x^2}{25}\right)^n$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x\cdot x^{2n}}{25\cdot 25^n} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{25^{n+1}} = \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{5^{2n+2}}}$$

converges when $0<\dfrac{x^2}{25}<1$

$0<\dfrac{x}{5}<1 \longrightarrow \boxed{0\le x<5}$

(b) $\dfrac{x^2}{(1-2x^2)^2}$ $= \dfrac{4x}{(1-2x^2)^2}\cdot\dfrac{x}{4}$

$$\left(\frac{1}{1-2x^2}\right)' = -\frac{1}{(1-2x^2)^2}(-4x) = \frac{4x}{(1-2x^2)^2}$$

$$\frac{1}{1-2x^2} = \sum_{n=0}^{\infty}(2x^2)^n = \sum_{n=0}^{\infty}2^n x^{2n}$$

$$\left(\frac{1}{1-2x^2}\right)' = \left(\sum_{n=0}^{\infty}2^n x^{2n}\right)' = \sum_{n=1}^{\infty}2^n\cdot(2n)\,x^{2n-1} = \sum_{n=1}^{\infty}2^{n+1}\cdot n\cdot x^{2n-1}$$

$$\frac{4x}{(1-2x^2)^2} = \sum_{n=1}^{\infty}2^{n+1}\cdot n\cdot x^{2n-1}$$

$$\frac{x^2}{(1-2x^2)^2} = \frac{x}{4}\cdot\frac{4x}{(1-2x^2)^2} = \frac{x}{4}\sum_{n=1}^{\infty}2^{n+1}n\,x^{2n-1} = \sum_{n=1}^{\infty}\frac{x}{4}2^{n+1}n\,x^{2n-1}$$

$$= \sum_{n=1}^{\infty}2^{n+1-2}\cdot n\cdot x^{2n-1+1}$$

$$= \boxed{\sum_{n=1}^{\infty}2^{n-1}\cdot n\cdot x^{2n}}$$

(c) $\ln(5-x)$

$$\int\frac{dx}{5-x} = -\ln(5-x)+C$$

$$\frac{1}{5-x} = \frac{1}{5}\frac{1}{1-\frac{x}{5}} = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$$

$$\ln(5-x) = -\int\frac{dx}{5-x}+C$$

$$\ln(5-x) = -\frac{1}{5}\int\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n dx$$

$$-\frac{1}{5}\sum_{n=0}^{\infty}\left(\int\frac{x^n}{5^n}dx\right) = -\frac{1}{5}\sum_{n=0}^{\infty}\left[\int\left(\frac{x}{5}\right)^n dx\right] = -\frac{1}{5}\sum_{n=0}^{\infty}\frac{1}{5^n}\frac{x^{n+1}}{n+1}+C$$

$$\ln(5-x) = C-\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^{n+1}\frac{1}{n+1}$$

plug in $x=0$: $\ln 5 = C+\sum_{n=0}^{\infty}0$

$C=\ln 5$

$$\boxed{\ln(5-x) = \ln 5-\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^{n+1}\frac{1}{n+1}}$$

(d) arctan(2x)

$\int \frac{du}{1+u^2} = \arctan u + C$ (if $u=2x$, then $du = 2dx$)

$\int \frac{2\,dx}{1+(2x)^2} = \arctan(2x) + C$

$$\frac{2}{1+(2x)^2} = 2\cdot\sum_{n=0}^{\infty}(-1)^n\left[(2x)^2\right]^n = 2\sum_{n=0}^{\infty}(-1)^n(2x)^{2n} = 2\sum_{n=0}^{\infty}(-1)^n \underbrace{2^{2n}\cdot x^{2n}}_{(2x)^{2n}}$$

$$\int \frac{2}{1+(2x)^2}\,dx = 2\sum_{n=0}^{\infty}(-1)^n 2^{2n}\left[\int x^{2n}\,dx\right]$$

$$\arctan(2x) = C + 2\sum_{n=0}^{\infty}(-1)^n 2^{2n}\cdot\frac{x^{2n+1}}{2n+1}$$

$$\arctan(2x) = C + \sum_{n=0}^{\infty}(-1)^n 2^{2n+1}\cdot\frac{x^{2n+1}}{2n+1}$$

plug into both sides $x=0$: $\arctan(0) = C \rightarrow \boxed{C=0}$

$$\boxed{\arctan(2x) = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1}\cdot\frac{x^{2n+1}}{2n+1}}$$

$|2x|<1$ – interval of convergence.

(e) $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$

$$\ln(1-x) = -\int\frac{dx}{1-x} = -\int\left(\sum_{n=0}^{\infty}x^n\right)dx = -\sum_{n=0}^{\infty}\left[\int x^n\,dx\right]$$

$$\ln(1-x) = C - \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

plug in $x=0$: $C = \ln 1 = 0$.

$$\boxed{\ln(1-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}}$$

$$\ln(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots\right]$$

$$\ln(1+x) = \int\frac{dx}{1+x} = \int\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)dx = \sum_{n=0}^{\infty}(-1)^n\left[\int x^n\,dx\right]$$

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} + C$$

plug in $x=0$: $C = \ln 1 = 0$

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\ln(1+x) - \ln(1-x) = \left(x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} + \dots\right) + \left(x + \cancel{\frac{x^2}{2}} + \frac{x^3}{3} + \cancel{\frac{x^4}{4}} + \dots\right)$$

$$= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \dots$$

$$\boxed{\ln\frac{1+x}{1-x} = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}}$$

3. Evaluate the indefinite integral.

(a) $\int \frac{x\,dx}{1+x^5}$

$$\frac{x}{1+x^5} = x\cdot\frac{1}{1+x^5} = x\cdot\sum_{n=0}^{\infty}(-1)^n (x^5)^n = x\sum_{n=0}^{\infty}(-1)^n x^{5n} = \sum_{n=0}(-1)^n x^{5n+1}$$

$$\int\frac{x\,dx}{1+x^5} = \int\left(\sum_{n=0}^{\infty}(-1)^n x^{5n+1}\right)dx = \sum_{n=0}(-1)^n\left[\int x^{5n+1}\,dx\right] = C + \sum_{n=0}(-1)^n\frac{x^{5n+2}}{5n+2}$$

(b) $\int \arctan(x^2)\,dx$

$\int\frac{du}{1+u^2} = \arctan u + C$, $\quad u = x^2$, $du = 2x\,dx$

$$\int\frac{2x\,dx}{1+(x^2)^2} = \arctan(x^2)+C \Rightarrow \int\frac{2x\,dx}{1+x^4} = \arctan(x^2)+C$$

$$\frac{2x}{1+x^4} = 2x\cdot\frac{1}{1+x^4} = 2x\cdot\sum_{n=0}^{\infty}(-1)^n(x^4)^n = 2x\sum_{n=0}^{\infty}(-1)^n x^{4n} = 2\sum_{n=0}^{\infty}(-1)^n x^{4n+1}$$

$$\arctan(x^2) = \int\frac{2x\,dx}{1+x^4} = 2\int\left(\sum_{n=0}^{\infty}(-1)^n x^{4n+1}\right)dx = 2\sum_{n=0}^{\infty}(-1)^n\left[\int x^{4n+1}\,dx\right] = 2\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{4n+2} + C$$

$$\arctan(x^2) = 2\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{4n+2} + C$$

$C = \arctan(0) = 0.$

$$\arctan(x^2) = 2\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{4n+2}$$

$$\int\arctan(x^2)\,dx = 2\int\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{4n+2}\right)dx$$

$$= 2\sum_{n=0}^{\infty}(-1)^n\frac{1}{4n+2}\left(\int x^{4n+2}\,dx\right)$$

$$\boxed{\int\arctan(x^2)\,dx = C + 2\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+3}}{(4n+2)(4n+3)}}$$

4. Approximate the value of the definite integral to six decimal places.

(a) $\int_0^{0.3}\frac{dx}{1+x^4}$

$$\frac{1}{1+x^4} = \sum_{n=0}^{\infty}(-1)^n(x^4)^n = \sum_{n=0}^{\infty}(-1)^n x^{4n}$$

$$\int_0^{0.3}\frac{dx}{1+x^4} = \sum_{n=0}^{\infty}(-1)^n\left(\int_0^{0.3}x^{4n}\,dx\right) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{4n+1}}{4n+1}\right)\Bigg|_0^{0.3}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(0.3)^{4n+1}}{4n+1} \approx \underset{(n=0)}{0.3} - \underset{n=1}{\frac{(0.3)^5}{5}} + \underset{n=2}{\frac{(0.3)^9}{9}}$$

$$\approx 0.3 - 0.000486 \approx \boxed{0.299514}$$

(b) $\int_0^{0.2} x\ln(1+x^2)\,dx$

Skip.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

5. Find the Taylor series representation for $f(x)$ centered at the given point.

(a) $f(x) = \ln x$, $a = 2$

$f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = +\frac{2}{x^3}$, ..., $f^{(n)}(x) = (-1)^n (n-1)!\, x^{-n} = \frac{(-1)^n (n-1)!}{x^n}$

$f^{(n)}(2) = \frac{(-1)^n (n-1)!}{2^n}$

$$\frac{f^{(n)}(2)}{n!} = \frac{(-1)^n (n-1)!}{n! \cdot 2^n} = \frac{(-1)^n}{n 2^n}$$

$$f(x) = \ln x = f(2) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n 2^n}(x-2)^n = \boxed{\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n 2^n}}$$

(b) $f(x) = e^{2x}$, $a = 3$

$f'(x) = 2e^{2x}$, $f''(x) = 4e^{2x}$, $f'''(x) = 8e^{2x}$, ..., $f^{(n)}(x) = 2^n \cdot e^{2x}$

$f^{(n)}(3) = 2^n \cdot e^6$

$$\frac{f^{(n)}(3)}{n!} = \frac{2^n e^6}{n!}$$

$$\boxed{e^{2x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{2^n e^6}{n!}(x-3)^n}$$

(c) $f(x) = \sin(2x)$, $a = \pi$

	$f(\pi) = \sin 2\pi = 0$
$f'(x) = 2\cos 2x$	$f'(\pi) = 2\cos 2\pi = 2$ $(k=0)$ $1 = 2\cdot 0+1$
$f''(x) = -4\sin 2x$	$f''(\pi) = -4\sin 2\pi = 0$
$f'''(x) = -8\cos 2x$	$f'''(\pi) = -8\cos 2\pi = -8$ $(k=1)$ $n = 3 = 2\cdot(1)+1$
$f^{iv}(x) = 16\sin 2x$	$f^{iv}(\pi) = 0$
$f^{v}(x) = 32\cos 2x$	$f^{v}(0) = 32$ $n=5 = 2\cdot 2+1$, $k=2$.

if n is even $n = 2k$

n is odd $n = 2k+1$

$$f^{(n)}(\pi) = \begin{cases} 0, & n = 2k \\ (-1)^k 2^{2k+1}, & n = 2k+1 \end{cases}$$

$$\boxed{\sin(2x) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} (x-2\pi)^{2k+1}}$$

(d) $f(x) = \sqrt{x}$, $a = 16$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

6. Find the Maclaurin series for $f(x)$.

(a) $f(x) = x\cos(2x)$

(b) $f(x) = xe^{-x^2}$

(c) $f(x) = \sqrt[3]{8+x}$