If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \times \mathbf{b}$, is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

The cross product $\mathbf{a} \times \mathbf{b}$ is a vector orthogonal to both vectors $\mathbf{a}$ and $\mathbf{b}$.

**Example 1.** *True/False.*

(a) *The cross product* $\mathbf{v} \times \mathbf{v} = 0$ *for any vector* $\mathbf{v}$.

*True* *False*

(b) *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are nonzero vectors. Then* $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$. *However,* $|\mathbf{a} \times \mathbf{b}| = |\mathbf{b} \times \mathbf{a}|$.

*True* *False*

(c) *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are nonzero vectors. Then* $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2$.

*True* *False*

$= a \times (a-b) + b \times (a-b)$

$= \cancel{a \times a}^{0} - a \times b + b \times a - \cancel{b \times b}^{0} = 2(b \times a).$

(d) *Consider a line given by parametric equations* $x + 2 = \dfrac{y-2}{3} = \dfrac{1-z}{5}$. *The vector* $\langle 1, 3, 5 \rangle$ *is a direction vector of the line.*

*True* *False*

$\Rightarrow \dfrac{x+2}{1} = \dfrac{y-2}{3} = \dfrac{z-1}{-5}$

$\Downarrow$

$\langle 1, 3, -5 \rangle$ is a direction vector.

(e) *Consider that a plane* $P_1$ *is orthogonal to the line given by equations*

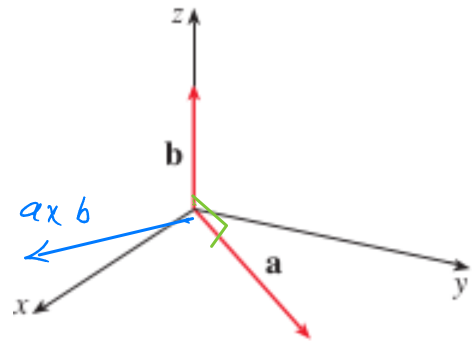
$$x = -1 + t,\ y = 2t,\ z = 3 - 3t.$$

*Then* $\mathbf{v} = \langle 2, 4, -6 \rangle$ *is a normal vector to the plane* $P_1$.

*True* *False*

$\rightarrow \langle 1, 2, -3 \rangle$ is a normal vector. Since $v$ is parallel to $\langle 1, 2, -3 \rangle$, $v$ is also a normal vector to the plane.

**Example 2** (12.4). *As shown in the figure below, let the vector* **a** *lies in the xy-plane and* **b** *is a vector in the direction of* **k**. *Suppose* $|\mathbf{a}| = 5$ *and* $|\mathbf{b}| = 4$.




*(I) Compute* $|\mathbf{a} \times \mathbf{b}|$ $= |a||b| \sin\theta$

$= (5)(4)(\sin 90^\circ)$

$= 20$ units.

*(II) Use the right hand rule to determine whether the components of* $\mathbf{a} \times \mathbf{b}$ *are positive, negative, or zero.*

Since $a \times b$ is $\perp$ to $b$, $a \times b$ lies on the xy-plane.

means z-component is zero.

By right hand rule, $a \times b$ should have y-component negative, and x-componet positive.

**Example 3** (12.4). *Consider a triangle ABC with vertices* $A(0,0,0)$, $B(1,2,0)$, *and* $C$ *on the* $y$*-axis. If the area of the triangle is* $3/2$, *determine the coordinates of the vertex* $C$.

$A(0,0,0)$, $B(1,2,0)$, $C(0,y,0)$.

$\overrightarrow{AB} = \langle 1, 2, 0 \rangle$, $\overrightarrow{AC} = \langle 0, y, 0 \rangle$

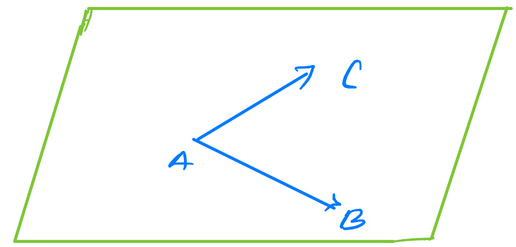
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & y & 0 \end{vmatrix} = 0i - 0j + yk = yk$$

Area of the triangle $= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{y^2}$

$\frac{3}{\cancel{2}} = \frac{1}{\cancel{2}}\sqrt{y^2} \Rightarrow y = \pm 3$

So, C is either $(0,3,0)$ or $(0,-3,0)$.

**Example 4** (12.4)**.** *Find unit vectors orthogonal to the plane passing through the points $A(1,0,0)$, $B(0,1,2)$ and $C(1,1,3)$.*




$\vec{AB} = \langle -1, 1, 2 \rangle$, $\vec{AC} = \langle 0, 1, 3 \rangle$

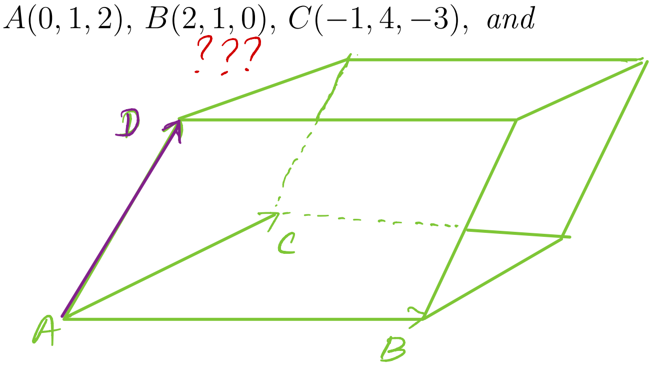
$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$

$= i + 3j - k$. So, $|\vec{AB} \times \vec{AC}| = \sqrt{1+9+1} = \sqrt{11}$

Unit normal vectors to the plane are

$$\vec{n} = \pm \frac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|} = \pm \frac{1}{\sqrt{11}} \langle 1, 3, -1 \rangle$$

**Example 5** (12.4)**.** *Determine whether the points $A(0,1,2)$, $B(2,1,0)$, $C(-1,4,-3)$, and $D(4,1,-2)$ lie in the same plane.*




$\vec{AB} = \langle 2, 0, -2 \rangle$

$\vec{AC} = \langle -1, 3, -5 \rangle$

$\vec{AD} = \langle 4, 0, -4 \rangle$

Volume of the parallelepiped $= |\vec{AB} \cdot (\vec{AC} \times \vec{AD})|$

$$= \begin{vmatrix} 2 & 0 & -2 \\ -1 & 3 & -5 \\ 4 & 0 & -4 \end{vmatrix}$$

$= 2(-12-0) - 0(4+20) - 2(0-12) = 0$

Since the volume of the parallelepiped is zero, the vectors must lie in the same plane $\Rightarrow$ the points must lie in the same plane.

*If $P$ and $Q$ are two points on a line $L$, then the* **vector equation of the line segment from $P$ to $Q$ is given by**

$$\langle x, y, z\rangle = \overrightarrow{OP} + t(\overrightarrow{PQ}) = (1-t)\overrightarrow{OP} + t\,\overrightarrow{OQ} \qquad 0 \le t \le 1$$

**Example 6** (12.5)**.** *Suppose the line $L_1$ passes through the point $P(2, 2, 5)$ and is parallel to the line*

$x_0 \quad y_0 \quad z_0$

$$L_2: \ \frac{x+5}{1} = \frac{y+1}{3} = \frac{z-2}{2}$$

A vector parallel to $L_2$ is $v = \langle 1, 3, 2\rangle$.

*(a) Determine symmetric equations of the line $L_1$.*

Direction vector of $L_2$ = direction vector of $L_1$ = $\langle \overset{a}{1}, \overset{b}{3}, \overset{c}{2}\rangle$.

$\langle x, y, z\rangle = \langle 2, 2, 5\rangle + t\langle 1, 3, 2\rangle$

$\Rightarrow \ x = 2 + t, \ y = 2 + 3t, \ z = 5 + 2t.$

*(b) Find the point of intersection of the line $L_1$ and the yz-plane (if any).*

yz-plane $\Rightarrow x = 0$.

so, $2 + t = 0 \Rightarrow t = -2$.

$y = 2 + 3(-2) = -4$

$z = 5 + 2(-2) = 1 \ \Rightarrow$ Intersection of $L_1$ and yz-plane is $(0, -4, 1)$.

*(c) Show that the line $L_1$ passes through the point $Q(3, 5, 7)$, and determine an equation of the line segment from $P$ to $Q$.*

$x = 3 \Rightarrow 2 + t = 3 \Rightarrow t = 1$

$y = 2 + 3(1) = 5, \quad z = 5 + 2(1) = 7.$

$\Rightarrow Q(3, 5, 7)$ is a point on $L_1$.

Equation of the line segment $PQ$ is

$\langle x, y, z\rangle = (1-t)\langle 2, 2, 5\rangle + t\langle 3, 5, 7\rangle, \quad 0 \le t \le 1$

$= \langle 2 - 2t + 3t, \ 2 - 2t + 5t, \ 5 - 5t + 7t\rangle$

$x = 2 + t, \ y = 2 + 3t, \ z = 5 + 2t; \ 0 \le t \le 1.$

**Example 7** (12.5)**.** *Determine whether the lines $L_1$ and $L_2$ are parallel, intersecting, or skew. If they intersect, find the point of intersection.*

direction vectors:

$$L_1 : \frac{x-1}{3} = \frac{y+3}{8} = z = t \qquad v_1 = \langle 3, 8, 1 \rangle$$

$$L_2 : \frac{x-3}{-2} = \frac{y-1}{-4} = \frac{z-4}{-4}. = s \qquad v_2 = \langle -2, -4, -4 \rangle$$

Since the direction vectors $v_1$ and $v_2$ are not parallel, the lines are NOT parallel.

Parametric forms:

$L_1$: $x = 1+3t$, $y = -3+8t$, $z = t$

$L_2$: $x = 3-2s$, $y = 1-4s$, $z = 4-4s$

Equating corresponding components;

$1+3t = 3-2s$, $-3+8t = 1-4s$, $t = 4-4s$

$-3 + 8(4-4s) = 1-4s$

$-3 + 32 - 32s = 1-4s \Rightarrow \boxed{s = 1}$

and $\boxed{t = 0}$

Substituting $s=1, t=0$ into $1+3t = 3-2s$;

$1+3(0) = 3-2(1)$

$1 = 1$ ✓

So, the lines intersect at a point.

The point of intersection: $t = 0 \Rightarrow x = 1, y = -3, z = 0$

OR

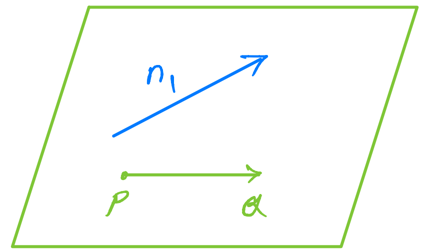
$s = 1 \Rightarrow x = 1, y = -3, z = 0$

The point of intersection of the lines is $(1, -3, 0)$.

**Example 8** (12.5). *Determine an equation of the plane passing through the points* $P(\overset{x_0}{1}, \overset{y_0}{1}, \overset{z_0}{3})$ *and* $Q(2, 1, 5)$ *and is perpendicular to the plane* $y = 2x + 3z - 4$. $\Rightarrow 2x - y + 3z = -4$

A normal to the plane $y = 2x + 3z - 4$ is $n_1 = \langle 2, -1, 3 \rangle$

$\vec{PQ} = \langle 1, 0, 2 \rangle$




A normal to the plane is

$$\vec{n} = \vec{PQ} \times \vec{n_1} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 2i + j - k = \langle \overset{a}{2}, \overset{b}{1}, \overset{c}{-1} \rangle$$

Equation of the plane is $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$$\langle 2, 1, -1 \rangle \cdot \langle x - 1, y - 1, z - 3 \rangle = 0$$

$-2 - 1 + 3$

$$2(x-1) + (y-1) - (z-3) = 0$$

$$2x + y - z = 0$$

> *The distance $D$ from the point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is*
>
> $$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

**Example 9** (12.5). *Find the distance between the parallel planes* $4x + 2y - z = 1$ *and* $8x + 4y - 2z = 7$.

A point on $4x + 2y - z = 1$ : $x = 0, y = 0, z = -1 \rightsquigarrow (\overset{x_1}{0}, \overset{y_1}{0}, \overset{z_1}{-1})$.

Distance betn the parallel planes = distance from $(0, 0, -1)$ to the plane $8x + 4y - 2z - 7 = 0$

$$= \frac{|8(0) + 4(0) - 2(-1) - 7|}{\sqrt{8^2 + 4^2 + (-2)^2}} = \frac{5}{\sqrt{84}} \text{ units.}$$

**Example 10** (12.5). *Consider two planes $3x - y + z = 2$ and $2x + y - z = 3$.*

*(a) Find the acute angle between these two planes.*

*(b) Find an equation of the line of intersection $L$ of these two planes.*

(a) A normal to $3x - y + z = 2$ is $n_1 = \langle 3, -1, 1 \rangle$

A normal to $2x + y - z = 3$ is $n_2 = \langle 2, 1, -1 \rangle$

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1|\,|n_2|} = \frac{6 - 1 - 1}{\sqrt{11}\,\sqrt{6}} = \frac{4}{\sqrt{66}} \Rightarrow \theta = \cos^{-1}\left(\frac{4}{\sqrt{66}}\right).$$

(b) $3x - y + z = 2 \Rightarrow z = 2 - 3x + y.$

Substituting into $2x + y - z = 3$ gives

$$2x + \cancel{y} - 2 + 3x - \cancel{y} = 3 \Rightarrow \boxed{x = 1}$$

Choosing $y = 0 \Rightarrow z = 2 - 3(1) + 0 = -1.$

So, $(x_0, y_0, z_0) = (1, 0, -1)$ is a point on the line of intersection.

A direction vector to $L_1$ is $v = \vec{n_1} \times \vec{n_2}$

$$v = \begin{vmatrix} i & j & k \\ 3 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 0i + 5j + 5k = \langle 0, 5, 5 \rangle$$

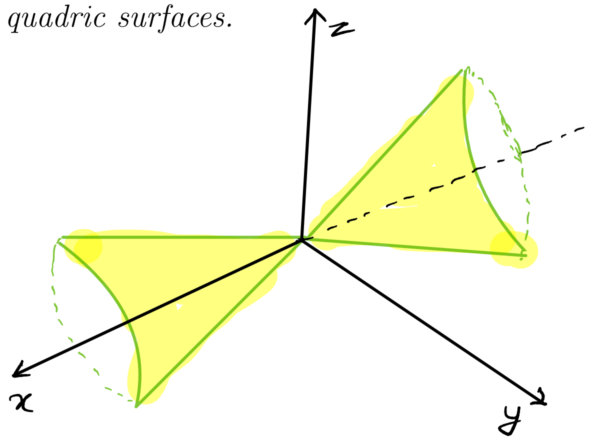
Eqn of the line: $\langle x, y, z \rangle = \langle 1, 0, -1 \rangle + t\langle 0, 5, 5 \rangle$

$$x = 1, \quad y = 5t, \quad z = -1 + 5t$$

**Example 11** (12.6)**.** *Identify and sketch the following quadric surfaces.*

*(a)* $x^2 = y^2 + z^2$

$x = \pm\sqrt{y^2 + z^2}$

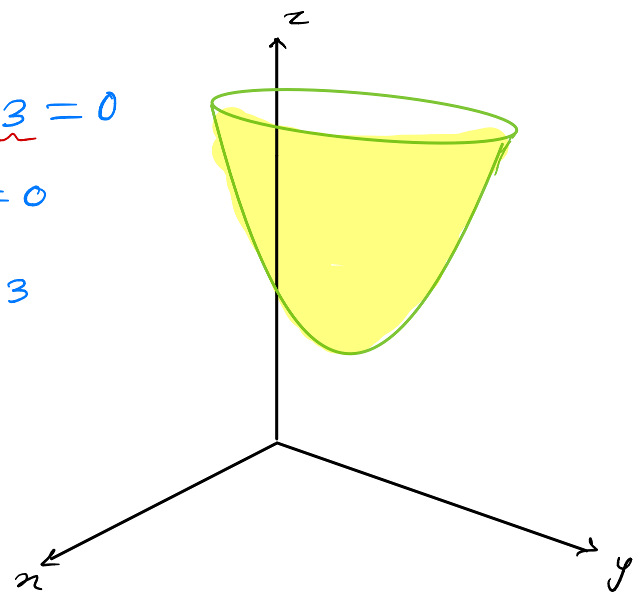



*(b)* $x^2 + y^2 - 2x - 2y - z + 5 = 0$

$x^2 - 2x + 1 + y^2 - 2y + 1 - z + 3 = 0$

$(x-1)^2 + (y-1)^2 - z + 3 = 0$

$\Rightarrow z = (x-1)^2 + (y-1)^2 + 3$




*(c)* $9x^2 - 9y^2 + z^2 - 9 = 0$

$x^2 - y^2 + \frac{z^2}{9} = 1$

$\Rightarrow \frac{x^2}{1} + \frac{z^2}{9} = 1 + y^2$

Choosing $y = k$, we get traces on $xz$-plane, which are ellipse of radius $\sqrt{1+k^2}$

In particular, $y = 0 \Rightarrow \frac{x^2}{1} + \frac{z^2}{9} = 1$

$y = \pm 3 \Rightarrow \frac{x^2}{4} + \frac{z^2}{36} = 1$