$\lim_{x\to 0} \sin x = 0$, $\lim_{x\to 0} \cos x = 1$, $\lim_{x\to 0} \frac{\sin x}{x} = 1$, $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$



1. Find the limit.

(a) $\lim\limits_{t\to 0} \frac{\sin^2 3t}{t^2} = \lim\limits_{t\to 0} \left(\frac{3\sin 3t}{3t}\right)^2$

$\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

$\lim\limits_{t\to 0} 3^2 \left(\frac{\sin 3t}{3t}\right)^2 = 9\left[\lim\limits_{t\to 0} \cancel{\frac{\sin 3t}{3t}}^{\,1}\right]^2 = \boxed{9}$

(b) $\lim\limits_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim\limits_{x\to 0} \frac{3\sin 3x}{3x} \cdot \frac{5x}{5\sin 5x} = \frac{3}{5} \lim\limits_{x\to 0} \cancel{\frac{\sin 3x}{3x}}^{\,1} \cancel{\frac{5x}{\sin 5x}}^{\,1} = \boxed{\frac{3}{5}}$

(c) $\lim\limits_{x\to 0} \frac{(\cos x - 1)\sin 3x}{x^2} = 3 \lim\limits_{x\to 0} \cancel{\frac{\cos x - 1}{x}}^{\,0} \cdot \cancel{\frac{\sin 3x}{3x}}^{\,1} = \boxed{0}$

(d) $\lim\limits_{x\to -2} \frac{\tan \pi x}{x+2}$

substitution $z = x+2$
$x = z-2$
$z \to 0$ as $x \to -2$

$= \lim\limits_{z\to 0} \frac{\tan[\pi(\underbrace{z-2}_{x})]}{z} = \lim\limits_{z\to 0} \frac{\overbrace{\tan(\pi z - 2\pi)}^{\tan \pi z}}{z}$

$= \lim\limits_{z\to 0} \frac{\tan \pi z}{z} = \lim\limits_{z\to 0} \frac{\sin \pi z}{z \cos \pi z} = \lim\limits_{z\to 0} \frac{\pi \sin \pi z}{\pi z} \cdot \cancel{\frac{1}{\cos \pi z}}^{\,1} = \pi \lim\limits_{z\to 0} \cancel{\frac{\sin \pi z}{\pi z}}^{\,1}$

$= \boxed{\pi}$

$(x^n)' = n x^{n-1}$, $([u(x)]^n)' = n[u(x)]^{n-1} u'(x)$ – the Chain Rule for power functions.

2. Differentiate the function.

(a) $s(t) = t^8 + 6t^7 - 18t^2 + 2t$

$s'(t) = (t^8)' + 6(t^7)' - 18(t^2)' + 2(t)' = \boxed{8t^7 + 42t^6 - 36t + 2}$

(b) $x(t) = \sqrt[3]{t} - \frac{1}{\sqrt[3]{t}} = t^{1/3} - t^{-1/3}$

$x'(t) = (t^{1/3})' - (t^{-1/3})' = \frac{1}{3} t^{1/3 - 1} - \left(-\frac{1}{3}\right) t^{-1/3 - 1} = \boxed{\frac{1}{3} t^{-2/3} + \frac{1}{3} t^{-4/3}}$

(c) $f(x) = (3x^3 - 2x^2 + 1)^6$

$f'(x) = 6(3x^3 - 2x^2 + 1)^5 (3x^3 - 2x^2 + 1)' = 6(3x^3 - 2x^2 + 1)^5 (9x^2 - 4x)$

(d) $G(x) = \frac{3x - 7}{x^2 + 5x - 4}$

$\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}}$

$G'(x) = \frac{(3x-7)'(x^2+5x-4) - (x^2+5x-4)'(3x-7)}{(x^2+5x-4)^2} = \frac{3(x^2+5x-4) + (2x+5)(3x-7)}{(x^2+5x-4)^2}$

$(\cos x)' = -\sin x$ $\quad (\tan x)' = \sec^2 x$ $\quad (\sec x)' = \sec x \tan x$

$(\sin x)' = \cos x$ $\quad (\cot x)' = -\csc^2 x$ $\quad (\csc x)' = -\csc x \cot x$

(e) $f(x) = \sqrt{x^3 - 3x^2 + 3x - 1}$ $= (x^3-3x^2+3x-1)^{1/2}$

$f'(x) = \frac{1}{2}(x^3-3x^2+3x-1)^{1/2-1}(x^3-3x^2+3x-1)' = \frac{1}{2}(x^3-3x^2+3x-1)^{-1/2}(3x^2-6x+3)$

(f) $g(\theta) = (1 + \cos^2\theta)^3$

$[f(g(x))]' = f'(g(x)) \cdot g'(x)$

$-\sin\theta$

$g'(\theta) = 3(1+\cos^2\theta)^2(1+\cos^2\theta)' = 3(1+\cos^2\theta)^2(2\cos\theta)(\cos\theta)'$

$= -3(1+\cos^2\theta)^2(2\cos\theta)(\sin\theta) = -6\cos\theta\sin\theta(1+\cos^2\theta)^2$

(g) $f(x) = \cos\sqrt{x}$

$f'(x) = -\sin\sqrt{x}\,(\sqrt{x})' = -\frac{\sin\sqrt{x}}{2\sqrt{x}}$

(h) $f(x) = \left(\frac{x^4-1}{x^4+1}\right)^3$

$$f'(x) = 3\left(\frac{x^4-1}{x^4+1}\right)^2 \left(\frac{x^4-1}{x^4+1}\right)'$$

$$= 3\left(\frac{x^4-1}{x^4+1}\right)^2 \frac{(x^4-1)'(x^4+1) - (x^4+1)'(x^4-1)}{(x^4+1)^2}$$

$$= 3\left(\frac{x^4-1}{x^4+1}\right)^2 \frac{4x^3(x^4+1) - 4x^3(x^4-1)}{(x^4+1)^2}$$

$$= 3\left(\frac{x^4-1}{x^4+1}\right)^2 \frac{4x^3\left[\cancel{x^4}+1 - (\cancel{x^4}-1)\right]}{(x^4+1)^2} = 3\left(\frac{x^4-1}{x^4+1}\right)^2 \frac{8x^3}{(x^4+1)^2}$$

$$= 24x^3 \frac{(x^4-1)^2}{(x^4+1)^4}$$

(i) $f(x) = \dfrac{2x+1}{\sqrt{x^2+3}} = \dfrac{2x+1}{(x^2+3)^{1/2}}$

$\frac{1}{2}(x^2+3)^{-1/2}(x^2+3)'$

$$f'(x) = \frac{\overset{2}{\cancel{(2x+1)'}}(x^2+3)^{1/2} - \cancel{\left[(x^2+3)^{1/2}\right]'}(2x+1)}{\left(\sqrt{x^2+3}\right)^2}$$

$$= \frac{2(x^2+3)^{1/2} - \frac{1}{2}(x^2+3)^{-1/2}\overset{2x}{\cancel{(x^2+3)'}}(2x+1)}{x^2+3}$$

$$= \frac{2(x^2+3)^{1/2} - \frac{1}{\cancel{2}}(x^2+3)^{-1/2}(\cancel{2}x)(2x+1)}{x^2+3}$$

$$= \frac{2(x^2+3)^{1/2} - x(2x+1)(x^2+3)^{-1/2}}{x^2+3}$$

$$[f(x)\cdot g(x)]' = f'(x)g(x) + g'(x)f(x)$$

(j) $f(x) = \underbrace{(x^6 + 4x^5 - 11)^5}\underbrace{(2 + x^8)^7}$

$$f'(x) = [(x^6+4x^5-11)^5]'(2+x^8)^7 + (x^6+4x^5-11)^5[(2+x^8)^7]'$$

$$= 5(x^6+4x^5-11)^4\underbrace{(x^6+4x^5-11)'}_{6x^5+20x^4}(2+x^8)^7 + (x^6+4x^5-11)^5\, 7(2+x^8)^6\underbrace{(2+x^8)'}_{8x^7}$$

$$= 5(x^6+4x^5-11)^4(6x^5+20x^4)(2+x^8)^7 + 56(x^6+4x^5-11)^5(2+x^8)^6(x^7)$$

3. Functions $f$ and $g$ satisfy the properties as shown in the table. Find the indicated quantity.

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | -3 | 3 | 1 | 1 |
| 2 | 0 | 3 | -5 | 10 |
| 3 | 2 | 5 | 0 | 4 |

(a) $h'(1)$, if $h(x) = f(g(x))$

(b) $z'(2)$, if $z(x) = [f(2x-1)]^4$

(c) $G'(1)$, if $G(x) = [x^2 - g(2x)]^3$

(a) $h(x) = f(g(x))$

$h'(x) = f'(g(x)) \cdot g'(x)$

$h(1) = f'(g(1)) \cdot g'(1)$ $\quad [\, g'(1) = 1,\ g(1) = 1 \,]$

$= f'(1) \cdot 1 = 3$

(b) $z(x) = [f(2x-1)]^4$

$z'(x) = 4\,[f(2x-1)]^3\,[f(2x-1)]'$

$= 4\,[f(2x-1)]^3\, f'(2x-1) \cdot (2x-1)'$

$= 4\,[f(2x-1)]^3\, f'(2x-1)\,(2)$

plug in $x=2$

$z'(2) = 4\,[f(4-1)]^3\, f'(4-1)\,(2)$

$= 8\,[f(3)]^3\, f'(3)$

$\boxed{f(3) = 2,\ f'(3) = 5}$

$= 8 \cdot 2^3 \cdot 5$

$= 8 \cdot 8 \cdot 5$

$= \boxed{320}$

(c) $G(x) = [x^2 - g(2x)]^3$

$G'(x) = 3\,[x^2 - g(2x)]^2\,(x^2 - g(2x))'$

$= 3\,[x^2 - g(2x)]^2\,(2x - g'(2x)\,(2x)')$

$= 3\,[x^2 - g(2x)]^2\,(2x - 2g'(2x))$ ← plug in $x=1$

$G'(1) = 3(1 - g(2))^2\,(2 - 2g'(2))$

$g(2) = -5,\ g'(2) = 10$

$= 3(1 - (-5))^2\,(2 - 2(10))$

$= 3(36)(-18) = \boxed{-1944}$

$ax^2+bx+c=0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

4. For what values of $x$ does the graph of $f(x) = 2x^3 - 3x^2 - 6x + 87$ have a horizontal tangent?

horizontal tangent $\Rightarrow$ slope $= 0$, $\Rightarrow$ slope $= f'(x)$

$f'(x) = (2x^3 - 3x^2 - 6x + 87)'$

$= 6x^2 - 6x - 6 = 0$

$x^2 - x - 1 = 0$

$x_1 = \frac{1+\sqrt{1+4}}{2} = \boxed{\frac{1+\sqrt{5}}{2}}$, $x_2 = \frac{1-\sqrt{1+4}}{2} = \boxed{\frac{1-\sqrt{5}}{2}}$

5. Find the equation of the tangent line to the curve $y = x\sqrt{1+x^2}$ at the point where $x = 1$.

Tangent line: $y - y(1) = y'(1)(x-1)$

$y(x) = x(1+x^2)^{1/2}$

$y(1) = 1(1+1)^{1/2} = \sqrt{2}$

$y'(x) = x'(1+x^2)^{1/2} + x\left[(1+x^2)^{1/2}\right]'$

$= (1+x^2)^{1/2} + x\,\frac{1}{2}(1+x^2)^{-1/2}(1+x^2)'$

$= (1+x^2)^{1/2} + \frac{x}{\cancel{2}}(1+x^2)^{-1/2}(\cancel{2}x)$

$y'(x) = (1+x^2)^{1/2} + \dfrac{x^2}{(1+x^2)^{1/2}}$

$y'(1) = (1+1)^{1/2} + \dfrac{1}{(1+1)^{1/2}} = \sqrt{2} + \dfrac{1}{\sqrt{2}} = \dfrac{2+1}{\sqrt{2}} = \dfrac{3(\sqrt{2})}{(\sqrt{2})(\sqrt{2})} = \dfrac{3\sqrt{2}}{2}$

tangent line: $\boxed{y - \sqrt{2} = \frac{3\sqrt{2}}{2}(x-1)}$

6. Find $\frac{dy}{dx}$ for the equation $\cos(x-y) = y\sin x$.

$$\frac{d}{dx}\left(\cos(x-y)\right) = \frac{d}{dx}\left(y\sin x\right)$$

$$-\sin(x-y)\,\frac{d}{dx}(x-y) = \frac{dy}{dx}\sin x + y\cos x$$

$$-\sin(x-y)\left(1-\frac{dy}{dx}\right) = \frac{dy}{dx}\sin x + y\cos x$$

solve for $\frac{dy}{dx}$:

$$-\sin(x-y) + \frac{dy}{dx}\sin(x-y) = \frac{dy}{dx}\sin x + y\cos x$$

$$\frac{dy}{dx}\sin(x-y) - \frac{dy}{dx}\sin x = y\cos x + \sin(x-y)$$

$$\frac{dy}{dx}\left(\sin(x-y) - \sin x\right) = y\cos x + \sin(x-y)$$

$$\boxed{\frac{dy}{dx} = \frac{y\cos x + \sin(x-y)}{\sin(x-y) - \sin x}}$$

7. Find $\dfrac{dx}{dy}$ for the equation $y^4 + x^2y^2 + yx^4 = y + 1$.

$$\frac{d}{dy}\left(y^4 + x^2y^2 + yx^4\right) = \frac{d}{dy}(y+1)$$

$$4y^3 + \frac{d}{dy}(x^2)\,y^2 + x^2\,\cancel{\frac{d}{dy}(y^2)}^{2y} + \cancel{\frac{d}{dy}(y)}^{1}\,x^4 + y\,\frac{d}{dy}(x^4) = \cancel{\frac{dy}{dy}}^{1}$$

$$4y^3 + 2x\frac{dx}{dy}\,y^2 + 2yx^2 + x^4 + y\,(4x^3)\frac{dx}{dy} = 1$$

$$2xy^2\frac{dx}{dy} + 4yx^3\frac{dx}{dy} = 1 - 4y^3 - 2yx^2 - x^4$$

$$\frac{dx}{dy}\left(2xy^2 + 4yx^3\right) = 1 - 4y^3 - 2yx^2 - x^4$$

$$\boxed{\frac{dx}{dy} = \frac{1 - 4y^3 - 2yx^2 - x^4}{2xy^2 + 4yx^3}}$$

8. Find the slope of the tangent line to the curve $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at the point (3,1).

slope = $\frac{dy}{dx}(3,1)$

$$2\frac{d}{dx}\left[(x^2+y^2)^2\right] = 25\frac{d}{dx}(x^2-y^2)$$

$$2(2)(x^2+y^2)\frac{d}{dx}(x^2+y^2) = 25\left(2x - 2y\frac{dy}{dx}\right)$$

$$\frac{4(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right)}{2} = \frac{50x - 50y\frac{dy}{dx}}{2}$$

$$4(x^2+y^2)\left(x+y\frac{dy}{dx}\right) = 25x - 25y\frac{dy}{dx}$$

$$4x(x^2+y^2) + 4y\frac{dy}{dx} = 25x - 25y\frac{dy}{dx}$$

$$4y\frac{dy}{dx} + 25y\frac{dy}{dx} = 25x - 4x(x^2+y^2)$$

$$29y\frac{dy}{dx} = 25x - 4x(x^2+y^2)$$

$$\boxed{\frac{dy}{dx} = \frac{25x - 4x(x^2+y^2)}{29y}}$$

$$\frac{dy}{dx}(3,1) \quad \frac{\text{plug in } x=3}{y=1} \quad \frac{25(3) - 4(3)(3^2+1)}{29(1)} = \frac{75-120}{29} = \boxed{-\frac{45}{29}}$$