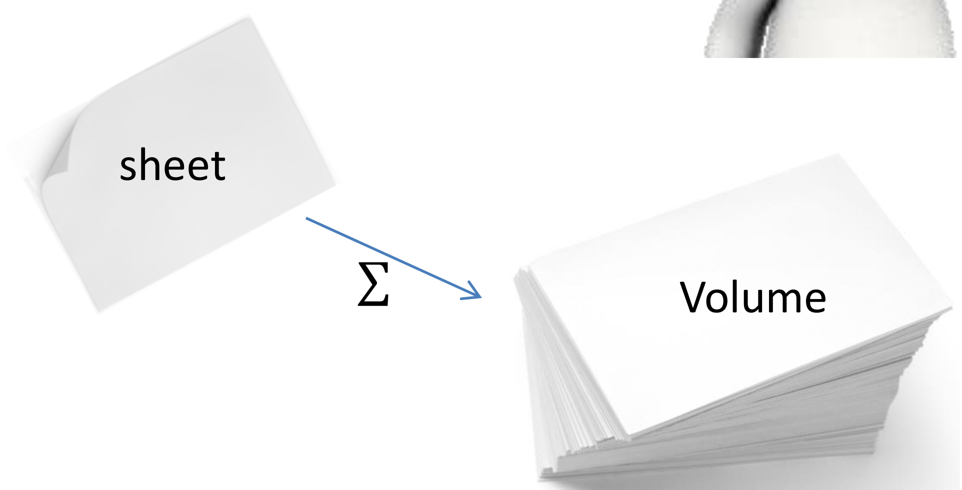
02

MATH 152
Week in Review

6.2 Volumes

6.3 Volume by Cylindrical Shells

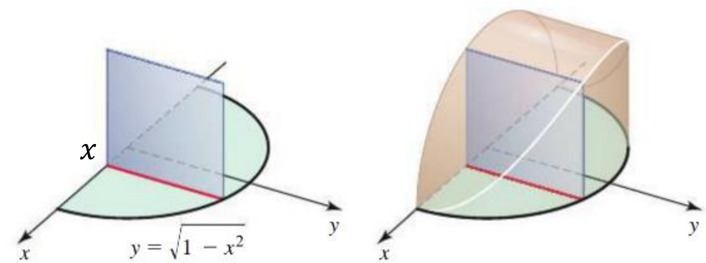

Reviews


sheet
Σ
Volume

General slicing method for any volume

The solid whose base is the region bounded by semicircle $y = \sqrt{1 - x^2}$ and the x −axis. And whose cross section through the solid perpendicular to the x axis are squares. Find the volume of the solid.

Step 1: Plot the graph

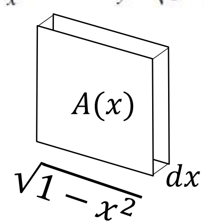


Step2. Find the size of a slice at x or y

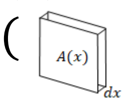
- At x: Thickness $dx \Rightarrow$ Cross section $A(x)$
- At y: Thickness $dy \Rightarrow$ Cross section $A(y)$

Step3. Find the volumnof a slice at x or y

- At x: Thickness $dx \Rightarrow$ Cross section $A(x)$
  - $dV = A(x)dx$
- At y: Thickness $dy \Rightarrow$ Cross section $A(y)$
  - $dV = A(y)dy$



$A(x) = \left[\sqrt{1 - x^2}\right]^2$

$V($  $) = A(x)dx$

$= (1 - x^2)dx$

Step4. Find the upper/lower limits for x or y

Step5. Set up integral and evaluate

$V = \int_a^b A(x)dx$ or $V = \int_c^d A(y)dy$

- Limit for x : $[-1,1]$
- $V = \int_a^b A(x)dx$

$= \int_{-1}^{1}(1 - x^2)dx = 2\int_0^1(1 - x^2)dx$

$= 2\left[x - \frac{1}{2x^2}\right]_0^1 = 1$

Volume by disks perpendicular to the x axis

Volume of solid of revolution around x axis

Step 1: Plot the graph

Step2. Find the size of a <u>perpendicular</u> slice at x

At x: Thickness $dx \Rightarrow$ Cross section $A(x)$

$$\pi[f(x)]^2$$

Step3. Find the volume of a slice at x

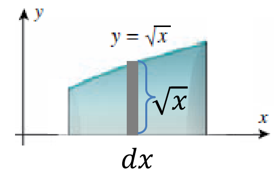
- At x: Thickness $dx \Rightarrow$ Cross section $A(x)$
  - $dV = \pi[f(x)]^2 dx$

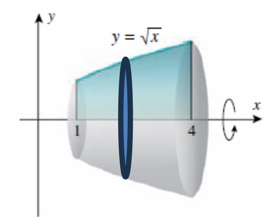
Step4. Find the upper/lower limits for x

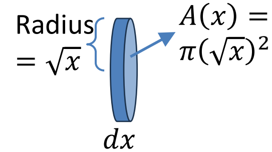
Step5. Set up integral and evaluate

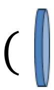
$$V = \int_a^b A(x)dx = \int_a^b \pi[f(x)]^2 dx$$

Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval [1, 4] is revolved about the x-axis







$V($  $) = \pi[f(x)]^2 dx$

$1 \le x \le 4$ (limits)

$$V = \int_a^b \pi[f(x)]^2 dx$$
$$= \int_1^4 \pi x dx$$
$$= \frac{\pi}{2}[x^2]_1^4$$
$$= \frac{\pi}{2}[16 - 1] = \frac{15}{2}\pi$$

Volume by Washer perpendicular to the x axis

Find the volume of the solid that is obtained when the region between the curve $y = \sqrt{x}$ *and* $y = 2$ over the interval [1, 4] is revolved about the *x*-axis

Volume of solid of revolution around x axis

Step 1: Plot the graph of $f(x)$, $g(x)$ w/ $f > g$

Step2. Find the size of a perpendicular slice at x

- At x: Thickness $dx \Rightarrow$ Cross section $A(x)$
- Washer = Large disc – small disc $= \pi[f(x)]^2 - \pi[g(x)]^2$

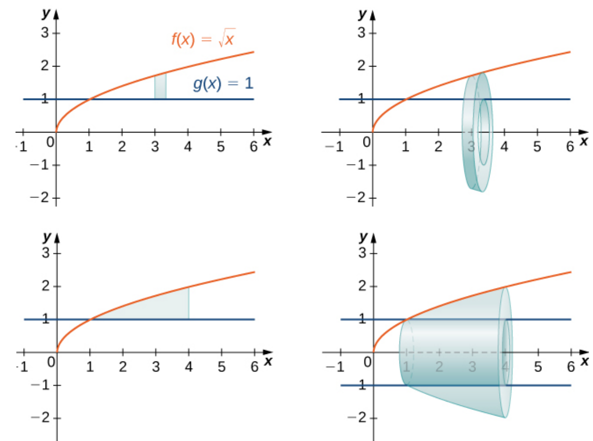
Step3. Find the volume of a slice at x

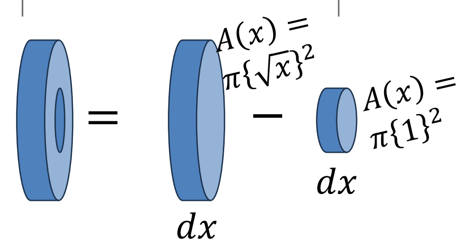
- At x: Thickness $dx \Rightarrow$ Cross section $A(x)$
  - $dV = \pi([f(x)]^2 - [g(x)]^2)dx$

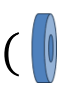
Step4. Find the upper/lower limits for x

Step5. Set up integral and evaluate

$$V = \int_a^b A(x)dx$$
$$= \int_a^b \pi([f(x)]^2 - [g(x)]^2)dx$$





$V($  $) = \pi\{\sqrt{x}\}^2 dx - \pi\{1\}^1 dx = \pi[x-1]dx$

Limit = [1,4]

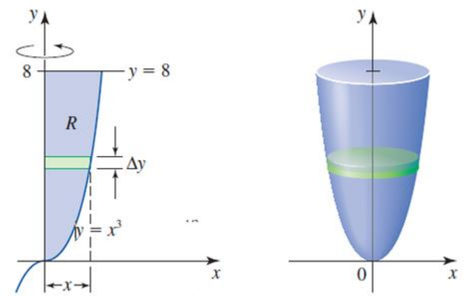
$$V = \int_1^4 \pi(x-1)dx$$
$$= \pi\left[\frac{1}{2}x^2 - x\right]_1^4 = \frac{9}{2}\pi$$

Volume by disks perpendicular to the *y* axis

Find the volume of the solid that is obtained when the region between the curve $y = x^3$ and $x = 0$ between the interval $0 \le y \le 8$ is revolved about the *y*-axis

Volume of solid of revolution around y axis

Step 1: Plot the graph



Step2. Find the size of a <u>perpendicular</u> slice at y

- At y: Thickness $dy \Rightarrow$ Cross section $A(y)$

$$\pi[g(y)]^2$$

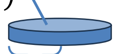
- For $y = f(x)$, solve for $x = f^{-1}(y)$

Step3. Find the volume of a slice at y

- At y: Thickness $dy \Rightarrow$ Cross section $A(y)$
  - $dV = \pi[g(y)]^2 dy$

dx

$A(x) = \pi(\sqrt[3]{y})^2$



dy

Radius $= \sqrt[3]{y}$

$V($  $) = \pi[g(y)]^2 dy$

Step4. Find the upper/lower limits for y

$0 \le y \le 8$ (limits)

Step5. Set up integral and evaluate

$$V = \int_c^d A(y)dy = \int_c^d \pi[g(y)]^2 dy$$

$$V = \int_a^b \pi[g(y)]^2 dy$$

$$= \int_0^8 \pi y^{\frac{2}{3}} dy$$

$$= \pi\left[\frac{3}{5}x^{\frac{5}{3}}\right]_0^8$$

Volume by Washer perpendicular to the y axis

Volume of solid of revolution around y axis

Step 1: Plot the graph of $f(y)$, $g(y)$ w/ $f > g$

Step2. Find the size of a perpendicular slice at y

- At y: Thickness $dy \Rightarrow$ Cross section $A(y)$
- Washer = Large disc – small disc

$$= \pi[f(y)]^2 - \pi[g(y)]^2$$

Step3. Find the volume of a slice at y

- At x: Thickness $dx \Rightarrow$ Cross section $A(y)$
    - $dV = \pi([f(y)]^2 - [g(y)]^2)dy$

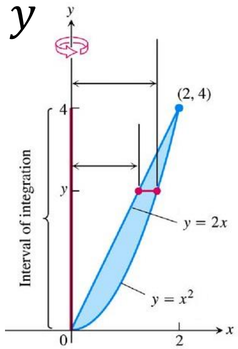
Step4. Find the upper/lower limits for y

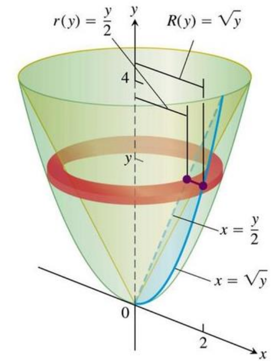
Step5. Set up integral and evaluate

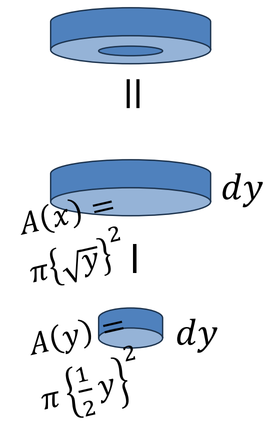
$$V = \int_a^b A(y)dy$$

$$= \int_a^b \pi([f(y)]^2 - [g(y)]^2)dy$$

Find the volume of the solid that is obtained when the region between the curve $y = x^2$ *and* $y = 2x$ is revolved about the *y*-axis







$V($$) = \pi\{\sqrt{x}\}^2 - \pi\{1\}^1 = \pi[x - 1]$

Limit = $[1,4]$

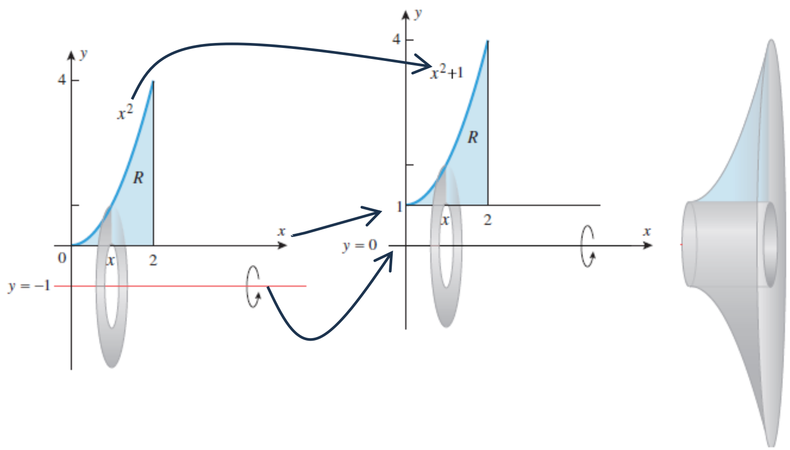
$$V = \int_1^4 \pi(x-1)dx$$

$$= \pi\left[\frac{1}{2}x^2 - x\right]_1^4 = \frac{9}{2}\pi$$

Other axis of revolution

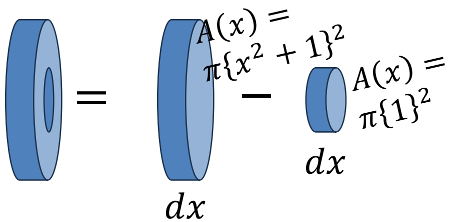
- Make the axis of revolution to $x-$ or $y-$ axis by translation
  - Translate the function and the axis of revolution
  - Align the axis of revolution to $x-$ or $y-$ axis

Example Find the volume of the solid generated when the region under the curve $y = x^2$ over the interval [0, 2] is rotated about the line $y = -1$.



- Translate 1 in y direction
  - $y = f(x) \Rightarrow y = f(x) + 1$
  - $\begin{cases} y = -1 & \Rightarrow & y = 0 \\ y = x^2 & \Rightarrow & y = x^2 + 1 \\ y = 0 & \Rightarrow & y = 1 \end{cases}$
  - $x = 2$ doesn't change by y −translation

Find the volume of the solid generated when the region between the curve $y = x^2 + 2$ and $y = 1$over the interval [0, 2] is rotated about the x axis.



$V($  $) = \pi\{x^2 + 1\}^2 dx - \pi\{1\}^1 = \pi[x^2 + 2x]dx$

Limit = [0,2]

$V = \int_0^2 \pi(x^2 + 2x)dx$

$= \pi\left[\frac{1}{3}x^3 - x^2\right]_0^2$

Volume by cylindrical shells about the y-axis

Volume of solid of revolution around y axis

Step 1: Plot the graph

Step2. Find the size of a parallel slice at x

At x: Thickness $dx \Rightarrow$ Cross section $A(x)$

$$2\pi f(x)dx$$

Step3. Find the volume of a slice at x

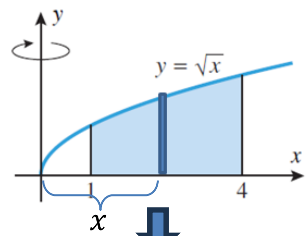
- At x: Thickness $dx \Rightarrow$ Cross section $A(x)$
  - $dV = 2\pi f(x)dx$

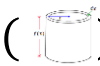
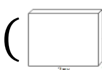
Step4. Find the upper/lower limits for x

Step5. Set up integral and evaluate

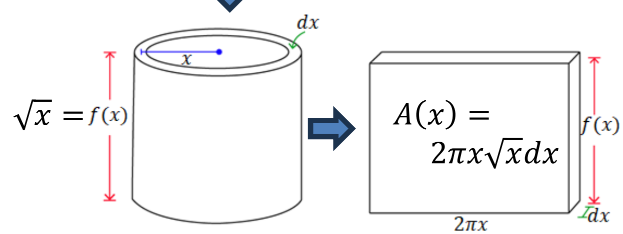
$V = \int_a^b A(x)dx = \int_a^b 2\pi x f(x)dx$

Use cylindrical shells to find the volume of the solid generated when the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$, and the x-axis is revolved about the y-axis.



$V($  $) = V($  $)$

$= 2\pi x(\sqrt{x})dx$



$$V = \int_a^b 2\pi f(x)dx$$
$$= \int_1^4 2\pi x\sqrt{x}dx$$

Volume by cylindrical shells about the x-axis

Volume of solid of revolution around x axis

Step 1: Plot the graph

Step2. Find the size of a <u>parallel</u> slice at y

At y: Thickness $dy \Rightarrow$ Cross section $A(y)$

$$2\pi f(y)dy$$

Step3. Find the volume of a slice at y

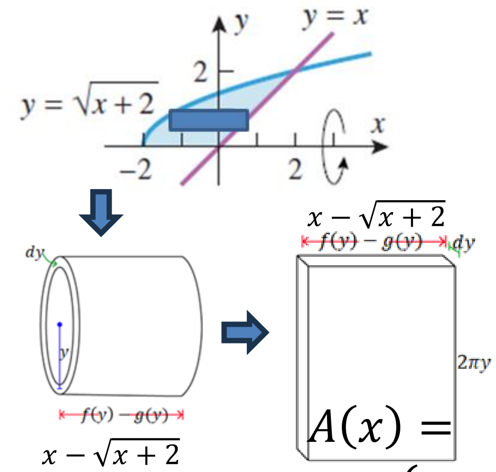
- At y: Thickness $dx \Rightarrow$ Cross section $A(y)$
  - $dV = 2\pi f(y)dy$

Step4. Find the upper/lower limits for y

Step5. Set up integral and evaluate

$$V = \int_a^b A(y)dy = \int_a^b 2\pi y f(y)dy$$

Use cylindrical shells to find the volume of the solid generated when the shaded region is revolved about the indicated axis.



$A(x) =$

$2\pi y\left(x - \sqrt{x-2}\right)dy$

$$V = \int_0^2 2\pi y[y - y^2 + 2]dy$$

$$= 2\pi \int_0^2 [y^2 - x^3 + 2y]dy$$

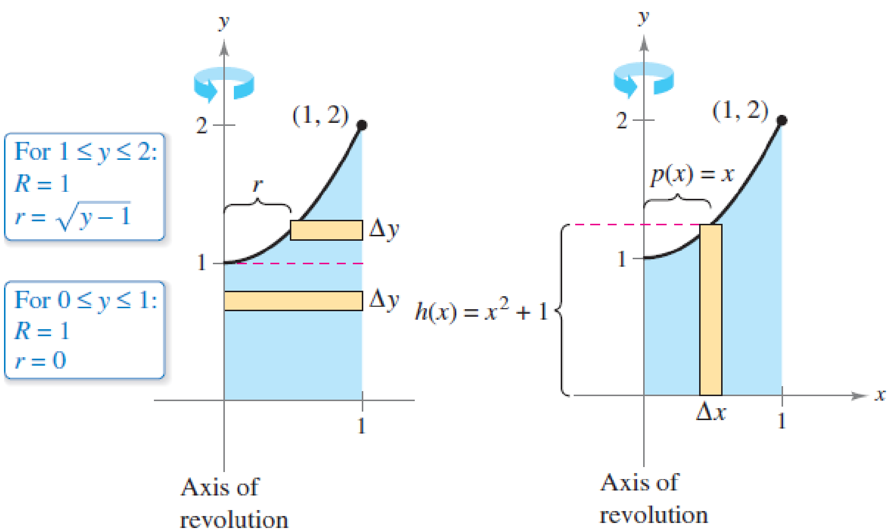
$$= 2\pi \left[\frac{1}{3}y^3 - \frac{1}{4}y^4 + y^2\right]_0^2$$

$$= 2\pi \left[\frac{8}{3} - \frac{16}{4} + 4\right]$$

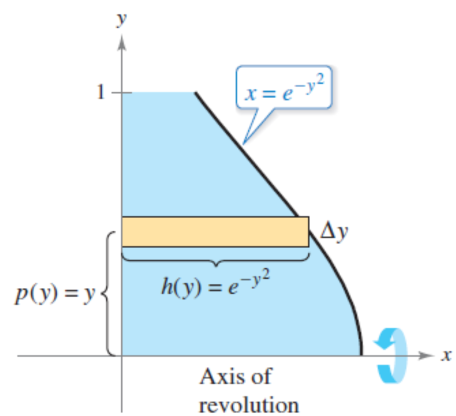
$$= \frac{16\pi}{3}$$

Comparison of Disk and Shell Methods

- Less integrals better (See if the volume can be found from a single integral)
- Try washer/disc methods first and see if the integral can be evaluated
  - See $\left(f(x)\right)^2$ or $\left(f^{-1}(y)\right)^2$ are integrable
- If the resulting integral is difficult to evaluate try the shell method
  - If the function is $\boldsymbol{y = f\left(ax^2 + b\right)}$ such as e^{-x^2} then the shell method is preferred : $\int 2\pi x f(ax^2 + b)dx$ w/ u-sub
- See if the integral is obtained without splitting regions with the shell method

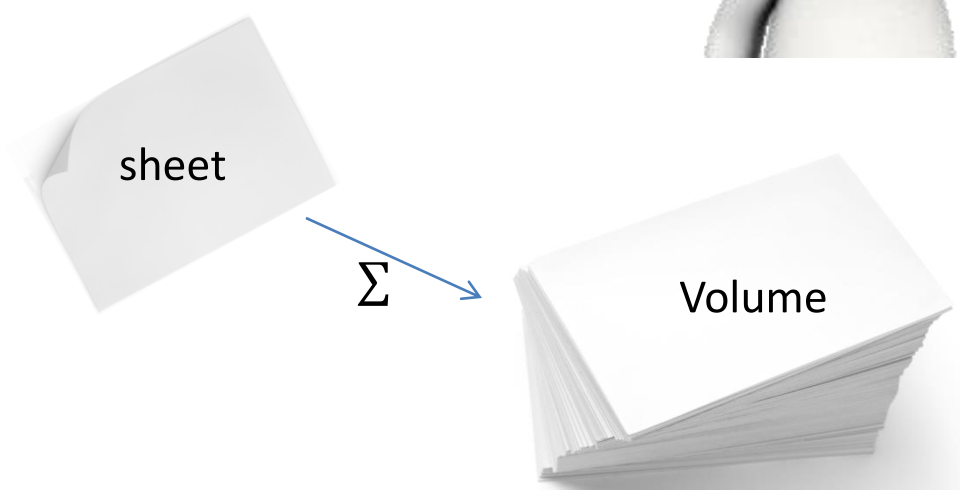


(a) Disk method (b) Shell method

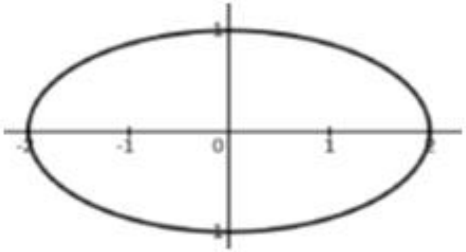


A shell method is preferred

Exercise



Find the volume of the solid whose base is the ellipse $x^2 + 4y^2 = 4$ and whose cross-sections perpendicular to the y-axis are squares. Evaluate your integral.



The region bounded by the curves $y = x^2$ and $y = 1$ is rotated about the line $y = 1$. Find the volume of the resulting solid.

(a) $\frac{8\pi}{15}$

(b) $\frac{8\pi}{5}$

(c) $\frac{4\pi}{3}$

(d) $\frac{12\pi}{5}$

(e) $\frac{16\pi}{15}$

The base of a solid is the region bounded by the curve $y = 5 - x^2$ and the x-axis. Cross-Sections perpendicular to the y-axis are rectangles with height equal to twice the base. Find the volume of this solid.

Consider the region R bounded by $y = 2x^2$ and $y = 1$, first quadrant only.
Find the volume obtained by rotating R about the y-axis.

(a) $\frac{\pi}{4}$

(b) $\frac{\pi}{2}$

(c) π

(d) $\frac{4\pi}{5}$

(e) None of the above

If we revolve the region bounded by $y = 1 - x^2$ and $x - y = 1$ about the line $y = 3$, which of the following integrals gives the resulting volume?

(a) $\int_{-1}^{2} 2\pi(3-x)(x^2-x+2)\,dx$

(b) $\int_{-2}^{1} \pi\left((2+x^2)^2-(4-x)^2\right)\,dx$

(c) $\int_{-1}^{2} 2\pi(x-3)(x^2-x+2)\,dx$

(d) $\int_{-2}^{1} \pi\left((4-x)^2-(2+x^2)^2\right)\,dx$

(e) $\int_{-1}^{2} \pi\left((2+x^2)^2-(4-x)^2\right)\,dx$

Consider the region R bounded by $y = \sqrt{x}$, $y = 1$, $x = 0$. Find the volume obtained by rotating the region R about the line $y = 1$.

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{2}$

(c) $\frac{7\pi}{6}$

(d) $\frac{\pi}{3}$

(e) $\frac{5\pi}{6}$

Consider the region R bounded by $y = 4x - x^2$ and $y = 0$. Which of the following integrals gives the volume of the solid obtained by revolving R about the line $x = -2$?

(a) $\int_{0}^{4} 2\pi(2-x)(4x-x^2)\,dx$

(b) $\int_{0}^{4} 2\pi x(4x-x^2)\,dx$

(c) $\int_{0}^{4} 2\pi(x+2)(4x-x^2)\,dx$

(d) $\int_{0}^{4} 2\pi(x-2)(4x-x^2)\,dx$

(e) None of the above

Find the volume of the solid found by rotating the region bounded by the curves $y = -x^2 + 2x$ and $y = 0$ about the y-axis.

(a) $\frac{16}{3}\pi$

(b) $\frac{8}{3}\pi$

(c) $\frac{4}{3}\pi$

(d) $\frac{2}{3}\pi$

(e) $\frac{1}{3}\pi$

Consider the solid S whose base is the region bounded by $y = 4 - x^2$ and $y = 0$. Cross sections perpendicular to the y - axis are semicircles. Find the volume of S.

Consider the solid S described here. The base of S is the region bounded by $y = x^2$ and $y = 4$. Cross sections perpendicular to the x - axis are squares. Find the volume of S.

Consider the region R bounded by $y = x^3$, $y = -x + 2$, $x = 0$, and $x = 1$.

(a) Sketch the region R.

(b) Set up the integral that gives the volume obtained by revolving the region R about the x-axis using the method of washers. **DO NOT EVALUATE THE INTEGRAL.**

(c) Set up the integral that gives the volume obtained by revoling the region R about the line $x = 1$ using the method of cylindrical shells. **DO NOT EVALUATE THE INTEGRAL.**

Consider the region R bounded by $y = \ln x$, $y = 0$, and $x = 2$. If this region is revolved about the line $y = -2$:

(a) Set up but **do not evaluate** the integral that gives the volume using the method of shells.

(b) Set up but **do not evaluate** the integral that gives the volume using the method of washers.

The region bounded by $y = \cos x$ and the x-axis on the interval $\left[0, \frac{\pi}{2}\right]$ is rotated about the x-axis. Find the volume of the resulting solid.

(a) 1

(b) $\frac{\pi^2}{2}$

(c) $\frac{\pi}{2}$

(d) $\frac{\pi}{4}$

(e) $\frac{\pi^2}{4}$ ← correct

The region bounded by $y = e^x$ and the x-axis on the interval $[0, 2]$ is rotated about the x-axis. Find the volume of the resulting solid.

(a) $\dfrac{\pi e^4}{2}$

(b) $\dfrac{\pi e^2}{2}$

(c) $\dfrac{\pi}{2}(e^4 - 1)$ ← correct

(d) $\dfrac{\pi}{2}(e^2 - 1)$

(e) $2\pi(e^4 - 1)$

Consider the region bounded by the curves $x = y^2 - 2y$ and the y-axis. Which of the following represents the volume of solid formed when the region is rotated about $y = 4$?

(a) $\displaystyle\int_0^2 2\pi y(y^2 - 2y)\, dy$

(b) $\displaystyle\int_0^2 2\pi y(2y - y^2)\, dy$

(c) $\displaystyle\int_0^2 2\pi(4 - y)(y^2 - 2y)\, dy$

(d) $\displaystyle\int_0^2 \pi(y - 4)(4y^2 - y^4)\, dy$

(e) $\displaystyle\int_0^2 2\pi(4 - y)(2y - y^2)\, dy$ ← correct

Consider the region bounded by the two curves $y = \cos x$, $y = \sin x$ and the two lines $x = 0$ and $x = \dfrac{\pi}{4}$. Which of the following represents the volume of this region being rotated about the line $x = -1$?

(a) $\displaystyle\int_0^{\frac{\pi}{4}} 2\pi(x+1)(\cos x - \sin x)\,dx$ ← correct

(b) $\displaystyle\int_0^{\frac{\pi}{4}} 2\pi(x+1)(\sin x - \cos x)\,dx$

(c) $\displaystyle\int_{-1}^{\frac{\pi}{4}} 2\pi(x+1)(\cos x - \sin x)\,dx$

(d) $\displaystyle\int_0^{\frac{\pi}{4}} 2\pi(x+1)(\cos^2 x - \sin^2 x)\,dx$

(e) $\displaystyle\int_0^{\frac{\pi}{4}} \pi(\cos^2 x - \sin^2 x)\,dx$

Which of the following integrals gives the volume of the solid obtained by rotating the region bounded by $y = 5 - x^2$ and $y = 1$ about the x-axis.

(a) $\displaystyle\pi\int_{-2}^{2} \left(1 - (5 - x^2)^2\right) dx$

(b) $\displaystyle\pi\int_{-2}^{2} (4 - x^2)^2\,dx$

(c) $\displaystyle 2\pi\int_{-2}^{2} x(4 - x^2)\,dx$

(d) $\displaystyle\pi\int_{-2}^{2} \left((5 - x^2)^2 - 1\right) dx$ ← correct

(e) $\displaystyle 2\pi\int_{-2}^{2} x(x^2 - 4)\,dx$

Find the volume of the solid obtained by rotating the region bounded by $x = y^2$ and $x = y^3$ around the y-axis.

(a) $\frac{\pi}{35}$

(b) $\frac{\pi}{10}$

(c) $\frac{\pi}{12}$

(d) $\frac{2\pi}{35}$ $\leftarrow$ correct

(e) $\frac{\pi}{105}$