6.2: SOLVING ODES WITH LAPLACE TRANSFORMS

Review

- Laplace transform of derivatives
  - $\mathcal{L}\{f'\} =$ $sF(s) - y(0)$
  - $\mathcal{L}\{f''\} =$ $s^2F(s) - sy(0) - y'(0)$
  - $\mathcal{L}\{f'''\} =$ $s^3F(s) - s^2y(0) - sy'(0) - y''(0)$
- How to solve differential equations with the Laplace transform
  - Laplace transform
  - Solve for $Y(s)$
  - Inverse transform

Exercise 1

Solve the initial value problem

$$y'' - 5y' + 6y = 0, \quad y(0) = -1, \quad y'(0) = 0.$$

Laplace transform

$$s^2 Y(s) - s\overset{-1}{\cancel{y(0)}} - \overset{0}{\cancel{y'(0)}} - 5\left(sY(s) - \overset{-1}{\cancel{y(0)}}\right) + 6Y(s) = 0$$

$$(s^2 - 5s + 6)Y(s) + s - 5 = 0$$

Solve for $Y(s)$

$$Y(s) = \frac{5-s}{s^2 - 5s + 6}$$

Inverse Laplace transform

$$Y(s) = \frac{5-s}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$$

$$5 - s = A(s-3) + B(s-2)$$

$s=2$: $\quad 3 = -A \Rightarrow A = -3$

$s=3$: $\quad 2 = B$

$$Y(s) = \frac{-3}{s-2} + \frac{2}{s-3}$$

$$\boxed{y(t) = -3e^{2t} + 2e^{3t}}$$

Exercise 2

Solve the initial value problem

$$y'' + 3y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 2.$$

Laplace transform

$$s^2 Y(s) - s\cancelto{0}{y(0)} - \cancelto{2}{y'(0)} + 3\left(sY(s) - \cancelto{0}{y(0)}\right) + 2Y(s) = 0$$

$$(s^2 + 3s + 2)Y(s) - 2 = 0$$

Solve for $Y(s)$

$$Y(s) = \frac{2}{s^2 + 3s + 2}$$

Inverse transform

$$Y(s) = \frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$2 = A(s+2) + B(s+1)$$

$s = -1$: $\quad 2 = -B \Rightarrow B = -2$

$s = -2$: $\quad 2 = A$

$$Y(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

$$\boxed{y(t) = 2e^{-t} - 2e^{-2t}}$$

Exercise 3

Solve the initial value problem

$$y'' + 4y' + 4y = 2e^t, \quad y(0) = 0, \quad y'(0) = 3.$$

Laplace transform

$$s^2 Y(s) - s\cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,3} + 4(sY(s) - \cancel{y(0)}^{\,0}) + 4Y(s) = 2\frac{1}{s-1}$$

$$(s^2+4s+4)Y(s) - 3 = \frac{2}{s-1}$$

Solve for $Y(s)$

$$(s^2+4s+4)Y(s) = 3 + \frac{2}{s-1}$$

$$Y(s) = \frac{3}{(s+2)^2} + \underbrace{\frac{2}{(s-1)(s+2)^2}}_{\text{do partial fractions}}$$

Inverse transform

$$\frac{2}{(s-1)(s+2)^2} = \frac{A}{s-1} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$

$$2 = A(s+2)^2 + B(s-1) + C(s-1)(s+2)$$

$s=1$: $\quad 2 = 9A \Rightarrow A = 2/9$

$s=-2$: $\quad 2 = -3B \Rightarrow B = -2/3$

$s=0:$ $$2 = 4A - B - 2C = 4\left(\tfrac{2}{9}\right) - \left(-\tfrac{2}{3}\right) - 2C$$

$$= \frac{8}{9} + \frac{2}{3} - 2C = \frac{14}{9} - 2C$$

$$\Rightarrow 2C = \frac{14}{9} - \frac{18}{9} = -\tfrac{4}{9}$$

$$C = -\tfrac{2}{9}$$

$$Y(s) = \frac{3}{(s+2)^2} + \frac{2}{9}\,\frac{1}{s-1} - \frac{2}{3}\,\frac{1}{(s+2)^2} - \frac{2}{9}\,\frac{1}{s+2}$$

$$\boxed{y(t) = 3te^{-2t} + \frac{2}{9}e^{t} - \frac{2}{3}te^{-2t} - \frac{2}{9}e^{-2t}}$$

Exercise 4

Solve the initial value problem

$$y''' + y' = 0, \quad y(0) = 0, \quad y'(0) = 4, \quad y''(0) = 2.$$

Laplace transform

$$s^3 Y(s) - s^2 \cancel{y(0)}^{0} - s\cancel{y'(0)}^{4} - \cancel{y''(0)}^{2} + (sY(s) - \cancel{y(0)}^{0}) = 0$$

$$(s^3 + s)Y(s) - 4s - 2 = 0$$

Solve for $Y(s)$

$$Y(s) = \frac{4s+2}{s^3+s}$$

Inverse transform

$$Y(s) = \frac{4s+2}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$4s + 2 = A(s^2+1) + (Bs+C)s$$

$$= As^2 + A + Bs^2 + Cs$$

$$= \underbrace{(A+B)}_{=0}s^2 + \underbrace{C}_{=4}s + \underbrace{A}_{=2}$$

$$B = -A = -2 \qquad C = 4 \qquad A = 2$$

$$Y(s) = \frac{2}{s} + \frac{-2s+4}{s^2+1}$$

$$= \frac{2}{s} - 2\,\frac{s}{s^2+1} + 4\,\frac{1}{s^2+1}$$

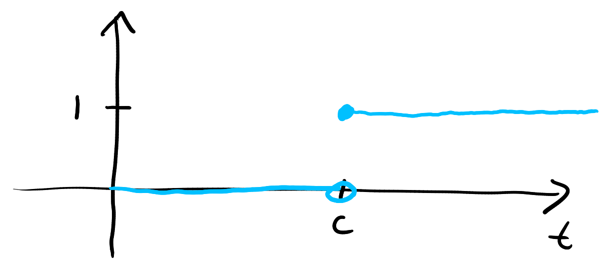
$$\boxed{y(t) = 2 - 2\cos(t) + 4\sin(t)}$$

6.3: STEP FUNCTIONS

Review

- The **unit step function** u_c is defined by

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$



- It can be used to write discontinuous functions into a single equation.
- The Laplace transform of u_c is

$$\mathcal{L}\{u_c\} = \frac{e^{-cs}}{s}$$

- Laplace transforms of **shifts**

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

$$\mathcal{L}\{u_c(t)f(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}$$

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$$

- Inverse Laplace transform of **shifts**

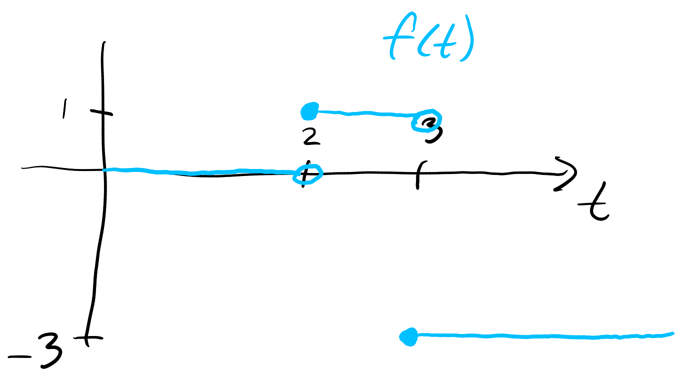
$$\mathcal{L}^{-1}\{F(s-c)\} = e^{ct}f(t)$$

Exercise 5

Convert the following function to a piecewise function. Also, graph the function. Compute its Laplace transform.

$$f(t) = u_2(t) - 4u_3(t)$$

$$f(t) = \begin{cases} 0 & t < 2 \\ 1 & 2 \le t < 3 \\ -3 & t \ge 3 \end{cases}$$



$$\mathcal{L}\{f\} = \mathcal{L}\{u_2\} - 4\mathcal{L}\{u_3\}$$

$$\boxed{= \frac{e^{-2s}}{s} - 4\frac{e^{-3s}}{s}}$$

Exercise 6

Convert the following function to a piecewise function. Compute its Laplace transform.

$$f(t) = t^2 - \sin(t-1)u_1(t) - t^2 u_2(t)$$

$$f(t) = \begin{cases} t^2 & t < 1 \\ t^2 - \sin(t-1) & 1 \le t < 2 \\ -\sin(t-1) & t \ge 2 \end{cases}$$

$$\mathcal{L}\{t^2 - \sin(t-1)u_1(t) - t^2 u_2(t)\}$$

$$= \frac{2}{s^3} - e^{-s}\mathcal{L}\{\sin(t+1-1)\} - e^{-2s}\mathcal{L}\{(t+2)^2\}$$

$$= \frac{2}{s^3} - e^{-s}\frac{1}{s^2+1} - e^{-2s}\mathcal{L}\{t^2+4t+4\}$$

$$\boxed{= \frac{2}{s^3} - e^{-s}\frac{1}{s^2+1} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)}$$

Exercise 7

Convert the following piecewise function into a form that involves step functions.

$$g(t) = \begin{cases} 0 & t < 3 \\ 5 & 3 \le t < 6 \\ \cos(2t) & t \ge 6 \end{cases}$$

$$g(t) = 5u_3(t) - 5u_6(t) + \cos(2t)u_6(t)$$

Exercise 8

Convert the following piecewise function into a form that involves step functions. Compute it's Laplace transform.

$$g(t) = \begin{cases} 2t & t < 4 \\ e^{2t} & t \ge 4 \end{cases}$$

$$g(t) = 2t - 2tu_4(t) + e^{2t}u_4(t)$$

$$G(s) = 2\frac{1}{s^2} - 2e^{-4s}\mathcal{L}\{t+4\} + e^{-4s}\mathcal{L}\{e^{2(t+4)}\}$$

$$= \frac{2}{s^2} - 2e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) + e^{-4s}e^{8}\mathcal{L}\{e^{2t}\}$$

$$\boxed{= \frac{2}{s^2} - 2e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) + e^{8}e^{-4s}\frac{1}{s+2}}$$

6.4: DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS FORCING FUNCTIONS

Exercise 9

Solve the initial value problem.

$$f'' + f = u_2(t), \quad f(0) = 0, \quad f'(0) = 0.$$

Laplace transform

$$s^2F(s) - s\cancel{f(0)}^{0} - \cancel{f'(0)}^{0} + F(s) = \frac{e^{-2s}}{s}$$

$$(s^2+1)F(s) = \frac{e^{-2s}}{s}$$

Solve for $F(s)$

$$F(s) = \frac{e^{-2s}}{s(s^2+1)}$$

Inverse transform

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)s$$

$$= As^2 + A + Bs^2 + Cs$$

$$= \underbrace{(A+B)}_{=0}s^2 + \underbrace{C}_{=0}s + \underbrace{A}_{=1}$$

$B = -A \qquad C = 0 \qquad A = 1$

$= -1$

$$F(s) = e^{-2s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$\boxed{f(t) = u_2(t)\left(1 - \cos(t-2)\right)}$$

Exercise 10

Solve the initial value problem.

$$w'' + w' = \begin{cases} 2 & t < 4 \\ 0 & t \geq 4 \end{cases}, \quad w(0) = 0, \quad w'(0) = 0.$$

$$= 2 - 2u_4(t)$$

Laplace transform

$$s^2 W(s) - s\cancel{w(0)}^{0} - \cancel{w'(0)}^{0} + sW(s) - \cancel{w(0)}^{0} = \frac{2}{s} - \frac{2e^{-4s}}{s}$$

$$(s^2+s)W(s) = (2-2e^{-4s})\frac{1}{s}$$

Solve for $W(s)$

$$W(s) = (2-2e^{-4s})\frac{1}{s(s^2+s)}$$

$$= (2-2e^{-4s})\frac{1}{s^2(s+1)}$$

Inverse transform

$$\frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$1 = A(s+1) + Bs(s+1) + Cs^2$$

$$= As + A + Bs^2 + Bs + Cs^2$$

$$= \underbrace{(B+C)}_{=0}s^2 + \underbrace{(A+B)}_{=0}s + \underbrace{A}_{=1}$$

$C = -B \qquad B = -A \qquad A = 1$

$\quad = 1 \qquad\quad = -1$

$$W(s) = (2-2e^{-4s})\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right)$$

$$= 2\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right) - 2e^{-4s}\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right)$$

$$\boxed{w(t) = 2t - 2 + e^{-t} - 2u_4(t)\left((t-4) - 1 + e^{-(t-4)}\right)}$$

Exercise 11

Consider a spring and mass system with a 4 kg mass hanging on a spring. When the mass is hung on the spring, the spring extends 40 cm. The mass experiences a damping force of 6 N when the mass is moving 3 m/s. The mass starts from equilibrium at rest, but there is an external force $\sin(t)$ that lasts for the first 2π seconds. Write down the initial value problem that describes this situation.

Find m, γ, k.

$m = 4\ kg$

$mg = kL \Rightarrow k = \frac{mg}{L} = \frac{4 \cdot 10}{0.4} = 100\ N/m$

$\gamma = \frac{|Force|}{speed} = \frac{6\ N}{3\ m/s} = 2\ \frac{N}{m/s}$

Initial value problem

$$\boxed{\begin{gathered} 4u'' + 2u' + 100u = \sin(t) - \sin(t)\, u_{2\pi}(t) \\ u(0) = 0, \quad u'(0) = 0. \end{gathered}}$$