01

MATH 152
Week in Review

5.5 The Substitution Rule

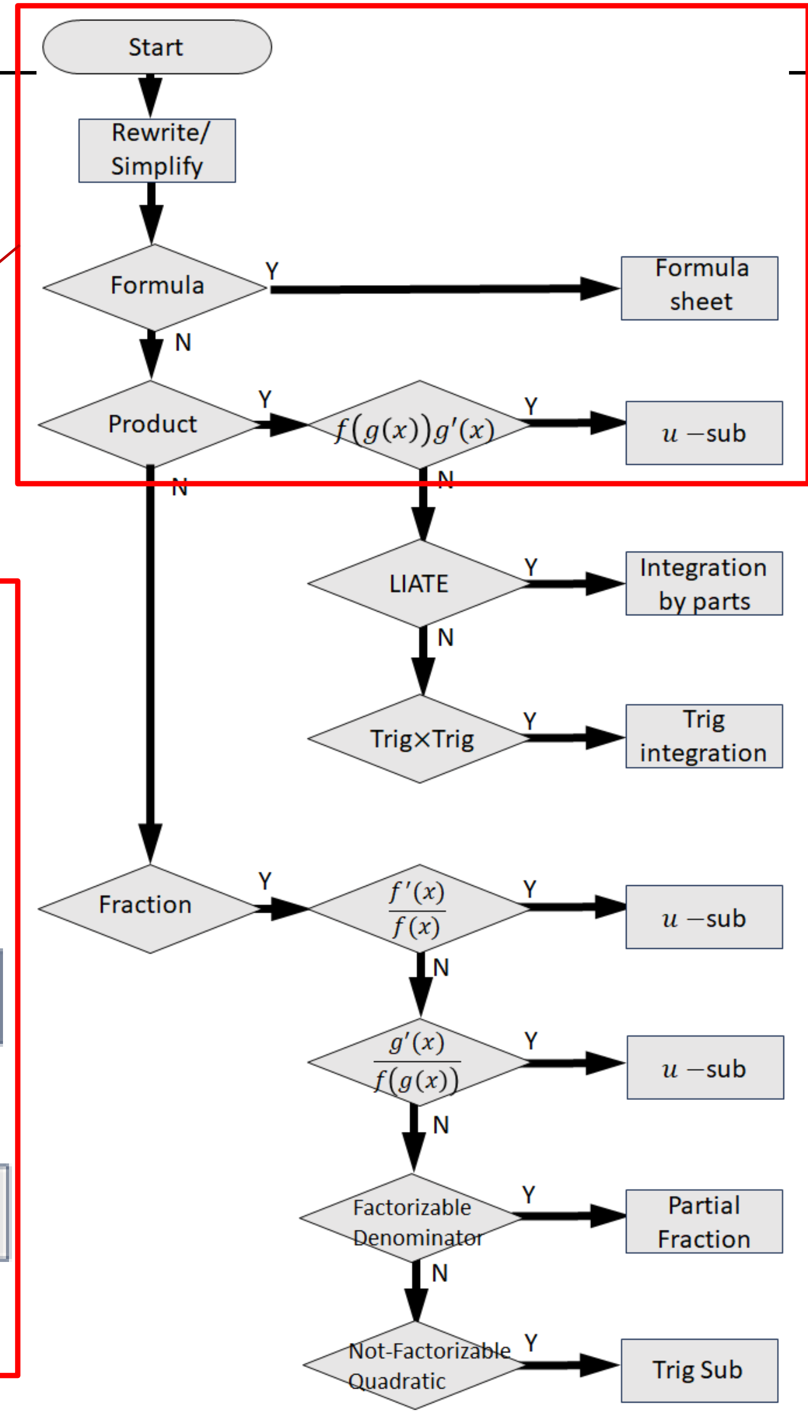
6.1 Area Between Curves

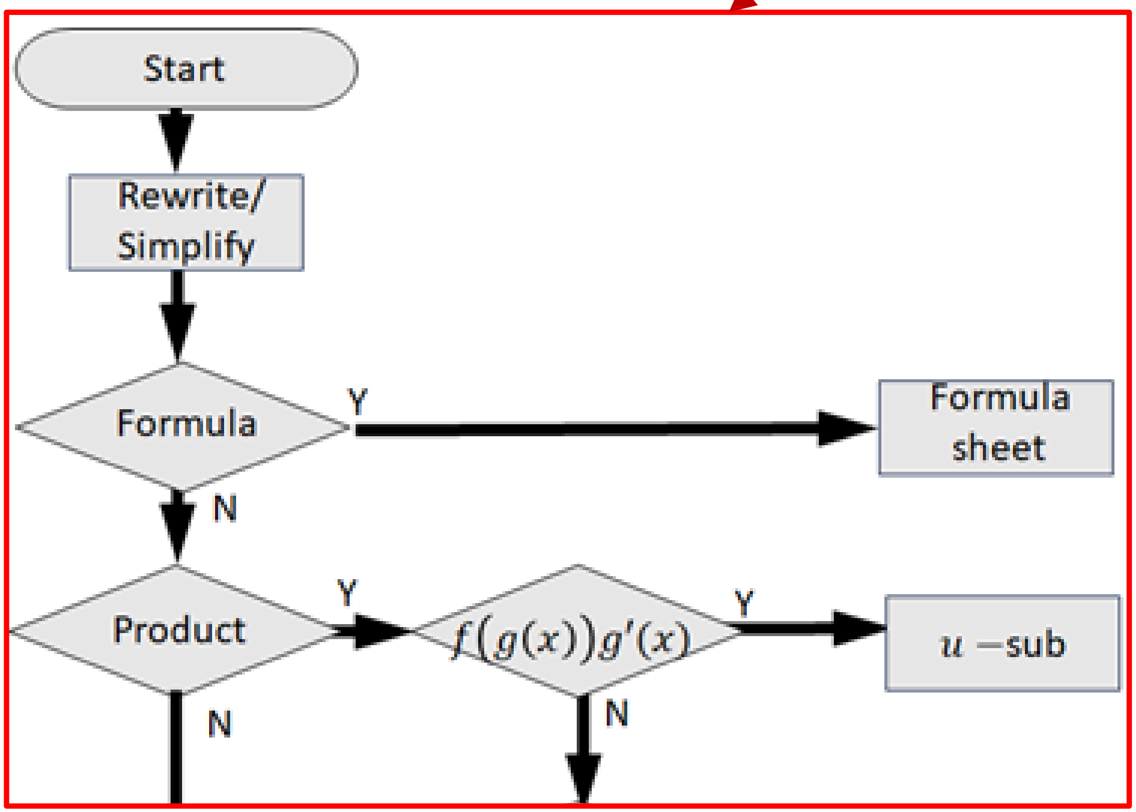
The Substitution Rule Review

Integration formulas

$\int 0\,dx =$	C
$\int dx =$	$x + C$
$\int x^r\,dx = \quad (r \neq -1)$	$\frac{x^{r+1}}{r+1} + C$
$\int \cos x\,dx =$	$\sin x + C$
$\int \sin x\,dx =$	$-\cos x + C$
$\int \sec^2 x\,dx =$	$\tan x + C$
$\int \sec x \tan x\,dx =$	$\sec x + C$
$\int \csc x \cot x\,dx =$	$-\csc x + C$
$\int e^x\,dx =$	$e^x + C$
$\int b^x\,dx = \ (0 < b, b \neq 1)$	$\frac{b^x}{\ln b} + C$
$\int \frac{1}{x}\,dx =$	$\ln\lvert x\rvert + C$
$\int \frac{1}{1+x^2}\,dx =$	$\tan^{-1} x + C$
$\int \frac{1}{\sqrt{1-x^2}}\,dx =$	$\sin^{-1} x + C$
$\int \frac{1}{x\sqrt{x^2-1}}\,dx =$	$\sec^{-1}\lvert x\rvert + C$

Integration Workflow

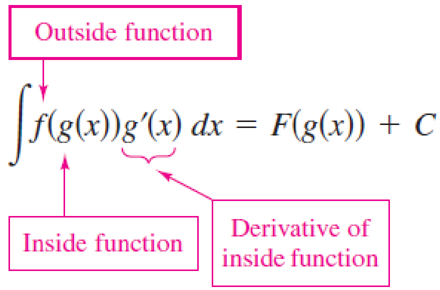




Method of u-substitution (Euler)

- Differentiation $\overleftrightarrow{\text{inverse}}$ Integration
  - In differentiation : Chain Rule : $\frac{d}{dx}\left[F\left(g(x)\right)\right] = F'\left(g(x)\right)g'(x)$
  - In integration: $\int F'\left(g(x)\right)g'(x)dx = F\left(g(x)\right) + C$

How to apply u −sub



1. Pattern recognition
- ***The integrand is a <u>product of two functions</u>***
  (composite fun * inner fun derivative)
  - f = outer function (usually more complex)
  - $g(x)$ = inner function (usually simpler)
2. Substitute the inner function as u
- $u = g(x)$ then $du = g'(x)dx$
- Include the constant term
3. Rewrite the integral in u: $\int f(u)du$
4. Integrate in u: $\int f(u)du = F(u) + C$
5. Substitute back to x: $\int f\left(g(x)\right)g'(x)dx = F\left(g(x)\right) + C$

$\int x^3\sqrt{2+x^4}\,dx$

$f = \quad \sqrt{2+x^4}$

$g = \quad 2+x^4$

$u = \quad 2+x^4$

$du = \quad 4x^3dx$

$x^3dx = \quad 1/4\,du$

$\int f(u)du =$

$\int \sqrt{\mathbf{2+x^4}}\,\mathbf{x^3dx} = \frac{1}{4}\int \sqrt{u}du$

$F(u) = \frac{1}{4}\left(\frac{2}{3}u^{3/2} + C\right)$

$F(x) = \frac{1}{6}(2+x^4)^{\frac{3}{2}} + C$

Change of Variables for Definite Integrals

- **Back substitution (Not recommended)**

If $\int_a^b f\big(g(x)\big)g'(x)dx = F\big(g(x)\big) + C$ then

$$\int_a^b f\big(g(x)\big)g'(x)dx = \big[F\big(g(x)\big)\big]_a^b = F\big(g(a)\big) - F\big(g(b)\big) = [F(u)]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)du$$

- **Complete substitution** $\int_a^b f\big(g(x)\big)g'(x)dx = \int_{g(a)}^{g(b)} f(u)\, du$
  - Substitute $u = g(x)$ then $du = g'(x)dx$
  - **Change the limits with respect to u**: $\int_a^b \rightarrow \int_{g(a)}^{g(b)}$ (g should be one to one)
  - Rewrite the integral in u: $\int_{g(a)}^{g(b)} f(u)\, du$
  - Integrate in u: $\int_{g(a)}^{g(b)} f(u)\, du = [F(u)]_{g(a)}^{g(b)} = F\big(g(b)\big) - F\big(g(a)\big)$

Example: Evaluate $\int_1^e \frac{\ln x}{x} dx$ by back substitution.

Find an antiderivative:

Rewrite : $\int \ln x \left(\frac{1}{x} dx\right)$

u-sub $u = \ln x\,;\, du = \frac{1}{x} dx$

Antiderivative $\int u du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$

Evaluate the integral: $\int_1^e \frac{\ln x}{x} dx = \left[\frac{1}{2}(\ln x)^2\right]_1^e = \frac{1}{2}(\ln e)^2 - \frac{1}{2}(\ln 1)^2$

$$= \frac{1}{2}(1) - \frac{1}{2}(0) = \frac{1}{2}$$

Examples

Evaluate $\int_1^e \frac{\ln x}{x}\, dx$ by complete substitution.

Rewrite : $\int \ln x \left(\frac{1}{x} dx\right)$

Id $f(g)$ and g'

- $f(g) =$ $\ln x \Rightarrow f(x) = x$
- $g =$ $\ln x$
- $g' =$ $1/x$

u-sub $u = \ln x \,;\, du = \frac{1}{x} dx$

complete substitution for the limits $\int_{x=1}^{x=e} \Rightarrow \int_{u=\ln 1}^{u=\ln e}$

Evaluate the integral:

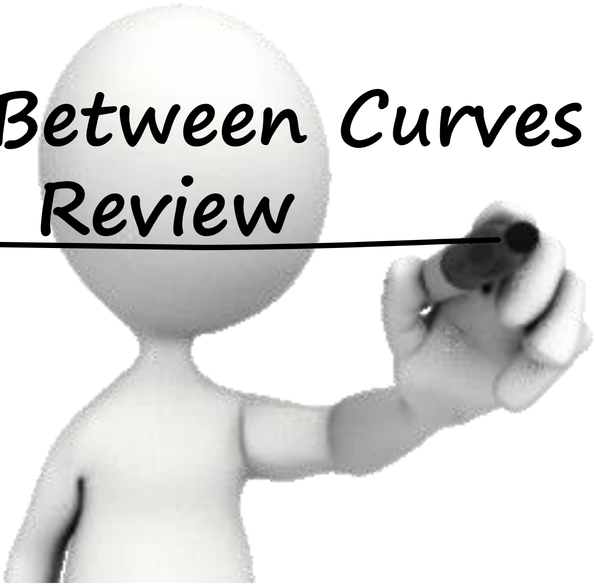
$$\int_1^e \ln x \left(\frac{1}{x} dx\right) = \int_{\ln 1}^{\ln e} u(du)$$

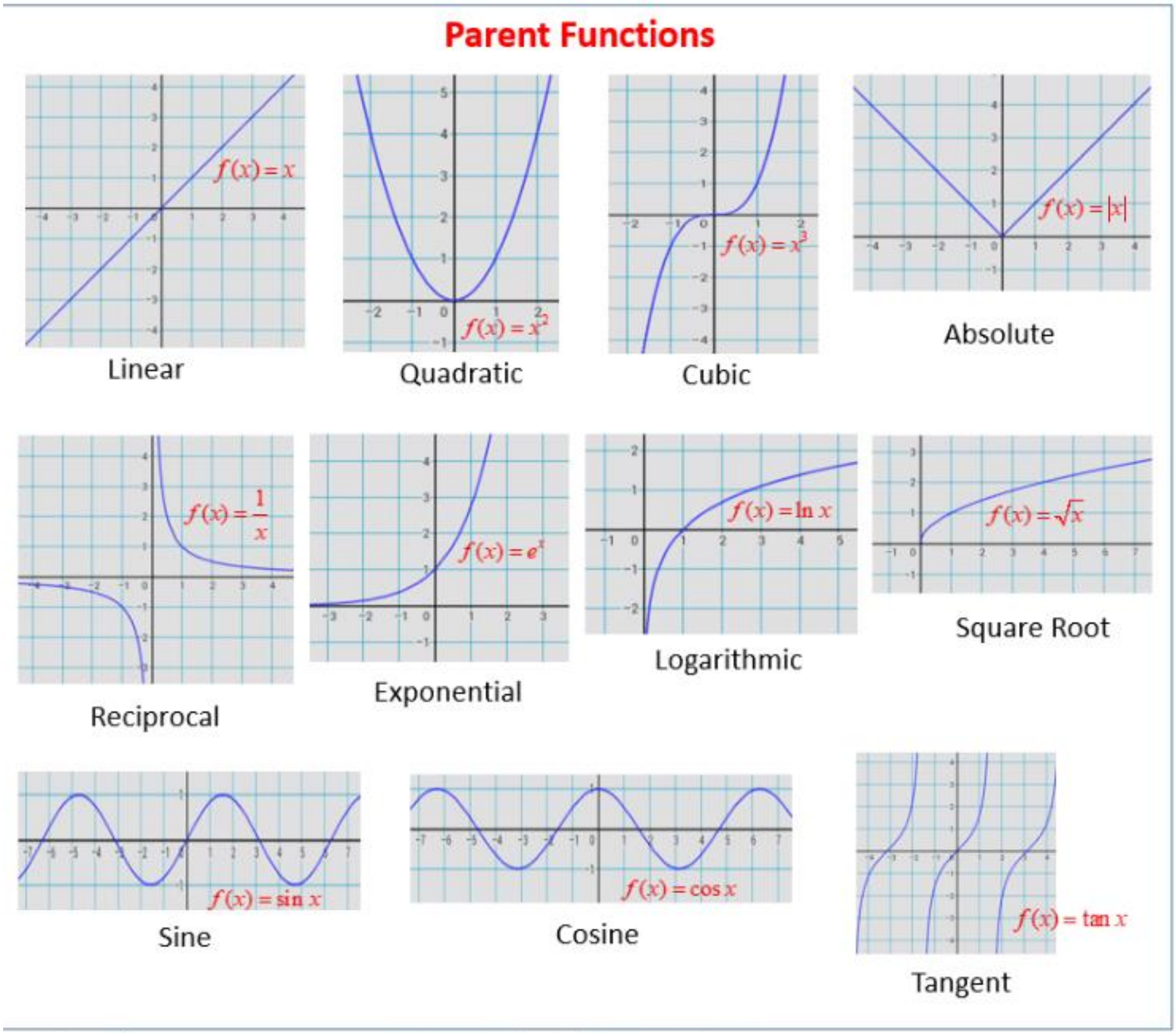
$$= \left[\frac{1}{2} u^2\right]_0^1$$

$$= \frac{1}{2}[1^2 - 0^2]$$

$$= \frac{1}{2}$$

Area Between Curves Review

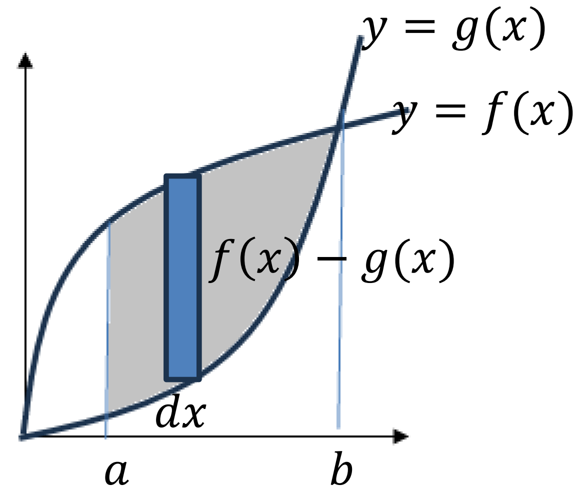



Parent Functions
f(x) = x
Linear
f(x) = x^2
Quadratic
f(x) = x^3
Cubic
f(x) = |x|
Absolute
f(x) = 1/x
Reciprocal
f(x) = e^x
Exponential
f(x) = ln x
Logarithmic
f(x) = √x
Square Root
f(x) = sin x
Sine
f(x) = cos x
Cosine
f(x) = tan x
Tangent

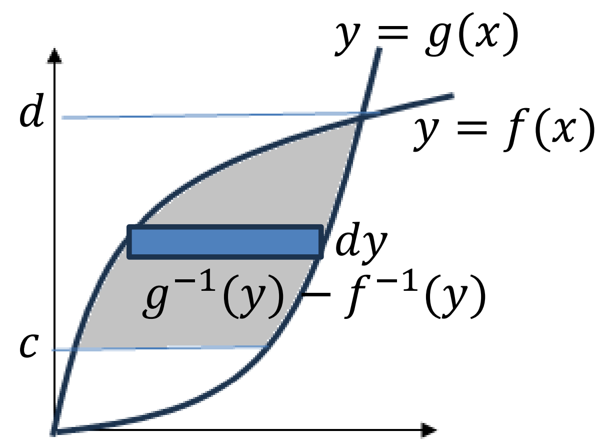
between curves $f(x)$ and $g(x)$, $f > g$

The integration domain is x axis $\Rightarrow dx$

1. Plot the graphs
2. Draw an infinitesimal strip



3. Find the area of the infinitesimal strip
   - [Function of x]dx
   - [Large fun – Small fun]dx
   - $[f(x) - g(x)]dx$
4. Find the upper/lower limits
   - Solve the equations if needed
   - In this case, $[a, b]$ is given
5. Integrate the area of infinitesimal strips
   - $\int_a^b [f(x) - g(x)]dx$



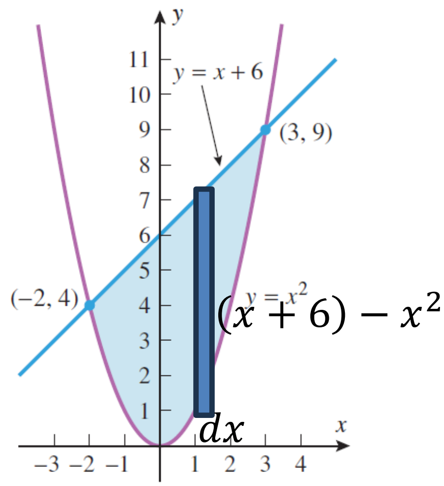
The integration domain is y axis $\Rightarrow dy$

2. the area of the infinitesimal strip
   - [Function of y]dy
   - [Large fun, g – Small fun,f]dy
   - $[g^{-1}(y) - f^{-1}(y)]dy$
   - $\int_c^d [g^{-1}(y) - f^{-1}(y)]dy$

How to find the area between curves $f(x)$ and $g(x)$

When $f(x) \geq g(x)$

1. Plot the graphs
2. Draw an infinitesimal strip
3. Find the area of the infinitesimal strip
4. Find the upper/lower limits
5. Integrate the area of infinitesimal strips



Area between the curves $y = x^2$ and $y = x + 6$.

$[(x+6) - x^2]dx$

(Need limits for x)

$x^2 = x + 6$

$(x+2)(x-3) = 0$

$\Rightarrow x = -2,3$

$= \int_{-2}^{3} (x + 6 - x^2)dx$

$= \left[\frac{1}{2}x^2 - 6x - \frac{1}{3}x^3\right]_{-2}^{3}$

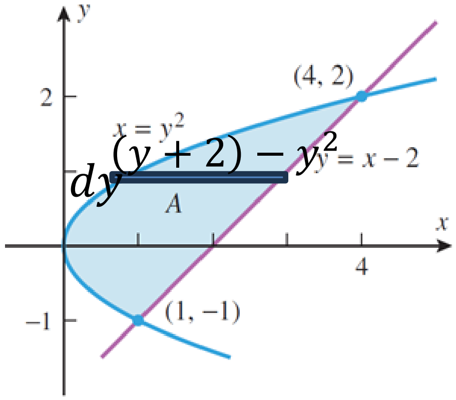
$= \frac{1}{2}[x^2]_{-2}^{3} - 6[x]_{-2}^{3} - \frac{1}{3}[x^3]_{-2}^{3}$

$= \frac{9-4}{2} - 6(3-(-2)) - \frac{27-(-8)}{3}$

$= -\frac{235}{6}$

How to find the area between curves $f(y)$ and $g(y)$

When $f(y) \geq g(y)$



Area between
$x = y^2$ and $y = x - 2$

1. Plot the graphs
2. Draw an infinitesimal strip
3. Find the area of the infinitesimal strip
4. Find the upper/lower limits
5. Integrate the area of infinitesimal strips

$[y + 2 - y^2]dy$

(Need limits for y)

- $y + 2 = y^2; y^2 - y - 2 = 0$

$$(y+1)(y-2) = 0$$
$$\Rightarrow y = -1, 2$$

$$\int_{-1}^{2} (y + 2 - y^2)\, dy$$
$$= \left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right]_{-1}^{2}$$
$$= \frac{1}{2}(2^2 - 1) + 2(2+1) - \frac{1}{3}(2^3 + 1)$$
$$= \frac{3}{2} + 6 - \frac{9}{3} = \frac{3}{2} + 3 = \frac{9}{2}$$

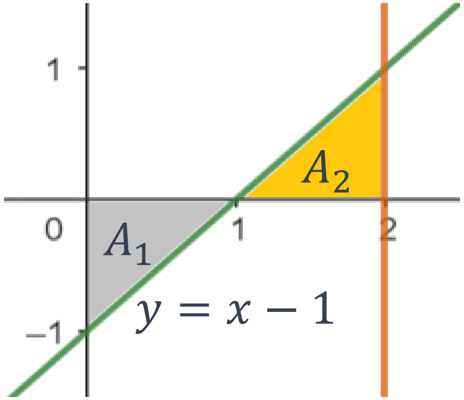
"Net signed area" vs "Total area"

Net signed area of $f(x)$ over (a,b) $= \int_a^b f(x)\,dx$

- Area taking into account both positive and negative regions of $f(x)$.
- The sign of the area indicates whether the curve lies above or below the reference axis.
- (Area of A_1) – (Area of A_2)

Total area of $f(x)$ over (a,b) $= \int_a^b |f(x)|\,dx$

- Area without considering whether the curve lies above or below the reference axis



Example:

Find the net area of $y = x - 1$ over $[0,2]$

$$A_1 = -\frac{1}{2} \text{ and } A_2 = \frac{1}{2}$$

$$\int_0^2 (x-1)dx = 0$$

Find the total area of $y = x - 1$ over $[0,2]$

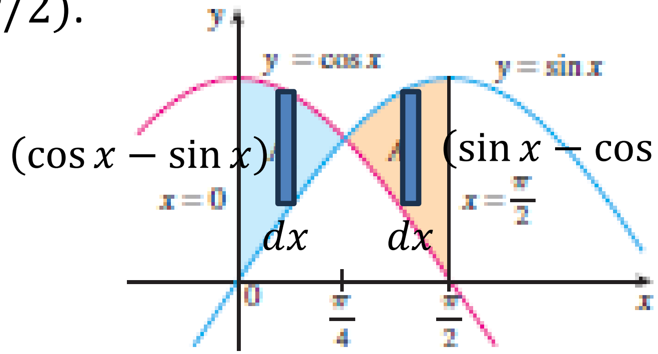
$$\int_0^2 |x-1|dx$$

$$= \int_0^1 \big(0 - (x-1)\big)dx + \int_0^1 \big((x-1) - 0\big)dx$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Area between curve = Total area

Area between the curves $y = \sin x, y = \cos x$ over $(0, \pi/2)$.



1. Plot the graphs
2. Separate the domain
3. Draw an infinitesimal strip
4. Find the area of the infinitesimal strip

$$\begin{cases} \{f(x) - g(x\}dx \text{ if } f > g \\ \{g(x) - f(x\}dx \text{ if } g > f \end{cases}$$

5. Find the upper/lower limits for each domains
6. Integrate the area of infinitesimal strips for each domains and add

On A_1, $(\cos x - \sin x)dx > 0$
On A_2, $(\sin x - \cos x)dx > 0$

$$\sin x = \cos x \quad \Rightarrow \frac{\sin x}{\cos x} = 1 \quad \Rightarrow \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} |\sin x - \cos x|\, dx$$

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= 2\sqrt{2} - 2$$

Net areas of odd/even functions over $[-a, a]$

$f(x)$ is even if $f(-x) = f(x)$

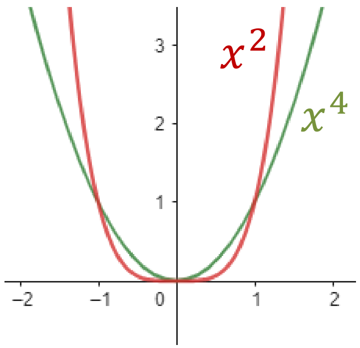
Example

$x^{even}, \cos x$

$f_{even} \pm g_{even}$

$f_{even} \times g_{even}$

$f_{odd} \times g_{odd}$



$\int_a^a (evev\ fun)dx$

$= 2\int_0^a (even\ fun)dx$

Example:

$\int_{-1}^{1}(3x^2)dx = [x^3]_{-1}^{1} = 1^3 - (-1)^3 = 2$

$2\int_0^1 (3x^2)dx = 2[x^3]_0^1 = 2(1-0) = 2$

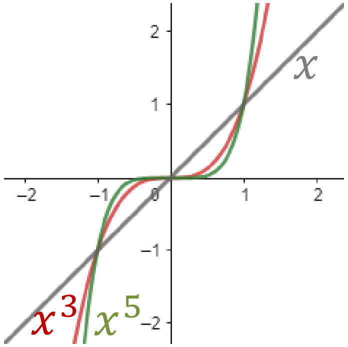
$f(x)$ is odd if $f(-x) = -f(x)$

Example

$x^{odd}, \sin x, \tan x$

$f_{odd} \pm g_{odd}$

$f_{even} \times g_{odd}$

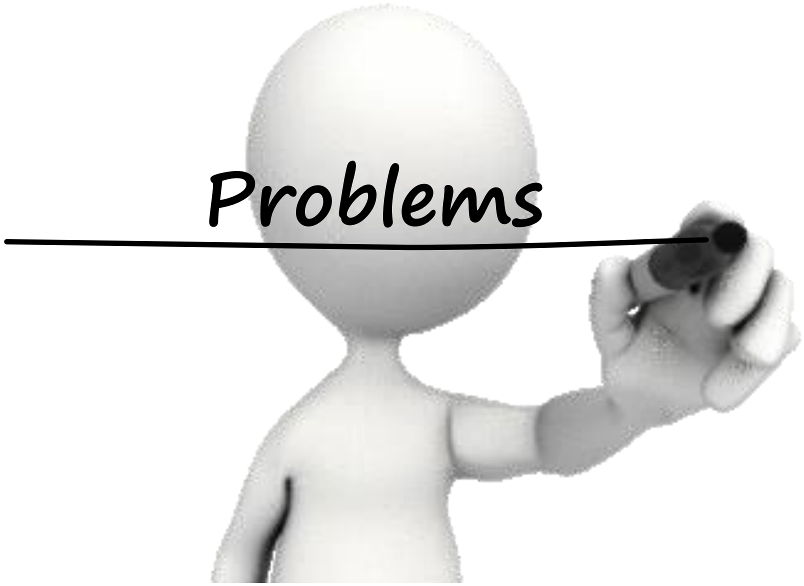


$\int_a^a (odd\ fun)dx$

$= 0$

Example:

$\int_{-1}^{1}(2x)dx = [x^2]_{-1}^{1} = 1^2 - (-1)^2 = 0$


Problems

Problems

Compute $\int x^3\sqrt{2+x^4}dx$

(a) None of these

(b) $\frac{1}{4}\left(2+x^4\right)^{3/2}+C$

(c) $\frac{1}{6}\left(2+x^4\right)^{3/2}+C$

(d) $\frac{8}{3}\left(2+x^4\right)^{3/2}+C$

(e) $\frac{3}{8}\left(2+x^4\right)^{3/2}+C$

Rewrite : $\int\sqrt{2+x^4}\left(\boldsymbol{x^3 dx}\right)$

Id f and g'

- $f =$ $\sqrt{g}$
- $g =$ $2+x^4$
- $g' =$ $4\boldsymbol{x^3}$

u-sub

$u = 2+x^4$

$du = 4\boldsymbol{x^3 dx}$

$\boldsymbol{x^3 dx} = \frac{1}{4}du$

Solve

$$\int\sqrt{2+x^4}\left(\boldsymbol{x^3 dx}\right)$$
$$= \int\sqrt{u}\left(\frac{1}{4}du\right)$$
$$= \frac{1}{4}\int u^{\frac{1}{2}}du$$
$$= \frac{1}{4}\left[\frac{2}{3}u^{\frac{3}{2}}\right]+C$$
$$= \frac{1}{6}(2+x^4)^{\frac{3}{2}}+C$$

Solution 2

Differentiate (a) to (e)

Problems

Compute $\int_0^{\sqrt{\pi}} x\sin(\pi - x^2)\,dx$

(a) $-\frac{\sin\sqrt{\pi}}{2}$

(b) -2

(c) -1

(d) 1

(e) 2

Rewrite : $\int_0^{\sqrt{\pi}} \sin(\pi - x^2)\,(xdx)$

Id f and g'

- $f =$
- $g =$
- $g' =$

$f = \sin x$
$g = \pi - x^2$
$g' = -2x$

u-sub

$u = \pi - x^2$
$du = -2xdx$
$\Rightarrow xdx = -\frac{1}{2}du$

complete substitution for the limits $\int_{x=0}^{x=\sqrt{\pi}} \Rightarrow \int_{\pi-0^2}^{\pi-\sqrt{\pi}^2}$

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

Evaluate the integral:

$$\int_\pi^0 \sin u\left(-\frac{1}{2}du\right)$$
$$= \frac{1}{2}\int_0^\pi \sin u\,du$$
$$= \frac{1}{2}[-\cos u]_0^\pi$$
$$= \frac{1}{2}[-\cos\pi + \cos 0] = 1$$

Problems

Evaluate $\displaystyle\int_0^{\pi/4} \frac{\sec^2(\theta)}{2+\tan(\theta)}\, d\theta$

(a) $\ln\left(\frac{4}{3}\right)$

(b) $\ln\left(\frac{\pi}{4}\right)$

(c) $\ln\left(\frac{\pi}{8}\right)$

(d) $\ln\left(\frac{\pi}{12}\right)$

(e) $\ln\left(\frac{3}{2}\right)$

Rewrite : $\int_0^{\pi/4} \frac{1}{2+\tan\theta}(\sec^2\theta\, d\theta)$

Id f and g'

- $f =$
- $g =$
- $g' =$

$f = 1/g$
$g = 2 + \tan\theta$
$g' = \sec^2\theta$

u-sub

$u = 2 + \tan\theta$
$du = \sec^2\theta\, d\theta$

complete substitution for the limits

$\int_{x=0}^{x=\pi/4} \Rightarrow \int_{2+\tan 0}^{2+\tan \pi/4}$

Evaluate the integral:

$\int_2^3 \frac{1}{u}(du)$
$= [\ln|u|]_2^3$
$= \ln 3 - \ln 2$
$= \ln\frac{3}{2}$

Problems

Evaluate $\int x^3\sqrt{x^2+1}\,dx$.

(a) $\frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$

(b) $\frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$

(c) $3x^2\sqrt{x^2+1} + \frac{x^4}{\sqrt{x^2+1}} + C$

(d) $\frac{2}{5}(x^2+1)^2 - \frac{2}{3}(x^2+1) + C$

(e) None of these

Rewrite : $\int x^2\sqrt{x^2+1}(xdx)$

Id f and g'

- $f =$
- $g =$
- $g' =$

$f = g\sqrt{g+1}$
$g = x^2$
$g' = 2xdx$

u-sub

$u = x^2+1 \Rightarrow x^2 = u-1$
$du = 2xdx$
$\Rightarrow xdx = \frac{1}{2}du$

Evaluate the integral:

$$\int x^2\sqrt{x^2+1}(xdx)$$
$$= \int (u-1)\sqrt{u}\left(\frac{1}{2}du\right)$$
$$= \frac{1}{2}\int\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$
$$= \frac{1}{2}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right] + C$$
$$= \frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1) + C$$

Problems

If f is continuous and $\int_0^{16} f(x)\,dx = 8$,

find $\int_0^4 x f(x^2)\,dx$.

(a) 16

(b) 2

(c) 8

(d) 64

(e) 4

Rewrite :

$\int_0^4 f(x^2)\,(xdx)$

Id f and g'

-
- $g =$
- $g' =$

$g = x^2$

$g' = 2x$

u-sub

$u = x^2$

$du = 2xdx$

$\Rightarrow xdx = \frac{1}{2}du$

complete substitution for the limits

$\int_{x=0}^{x=4} \Rightarrow \int_{u=0^2}^{u=4^2}$

Evaluate the integral:

$\int_0^{16} f(u)\left(\frac{1}{2}du\right)$

$= \frac{1}{2}\int_0^{16} f(u)du$

$= \frac{1}{2}(8)$

$= 4$

Problems

Find the area bounded by
$y = e^x$, $y = e^{-x}$, $x = 0$, and $x = 1$.

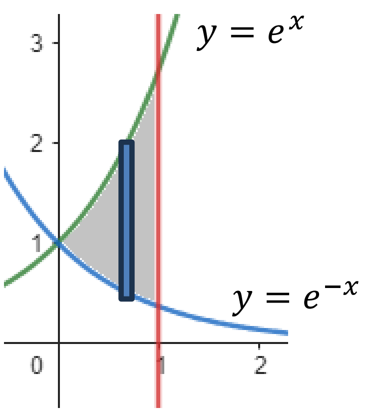
(a) $e + \frac{1}{e} - 2$

(b) $e - \frac{1}{e}$

(c) $e + \frac{1}{e} + 2$

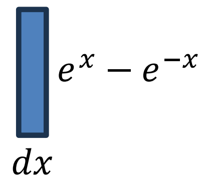
(d) $1 + \frac{1}{e}$

(e) $1 + \frac{1}{e} - 2$



When $f(y) \geq g(y)$

1. Plot the graphs
2. Draw an infinitesimal strip
3. Find the area of the infinitesimal strip

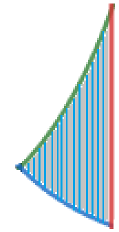


$A(|) = (e^x - e^{-x})dx$

4. Find the upper/lower limits

$0 \leq x \leq 1$

5. Integrate the area of infinitesimal strips



$$\int_0^1 (e^x - e^{-x})dx$$
$$= [e^x + e^{-x}]_0^1$$
$$= (e^1 + e^{-1}) - (1 + 1)$$
$$= e + \frac{1}{e} - 2$$

Problems

Which of the following represents the area bounded by the curves $y = x^2 - 2x$ and $y = 2x$ on the interval from $x = 1$ to $x = 6$?

(a) $\int_0^6 4x - x^2 \, dx$

(b) $\int_0^4 4x - x^2 \, dx$

(c) $\int_1^4 x^2 - 4x \, dx + \int_4^6 4x - x^2 \, dx$

(d) $\int_1^6 4x - x^2 \, dx$

(e) $\int_1^4 4x - x^2 \, dx + \int_4^6 x^2 - 4x \, dx$

1. Plot the graphs

2. Draw an infinitesimal strip

3. Find the area of the infinitesimal strip

   $x^2 - 2x - 2x$, dx: $A(|) = (x^2 - 4x)dx$

   $2x - (x^2 - 2x)$, dx: $A(|) = (4x - x^2)dx$

4. Find the upper/lower limits

   $x^2 - 2x = 2x \Rightarrow x^2 - 4x = 0$

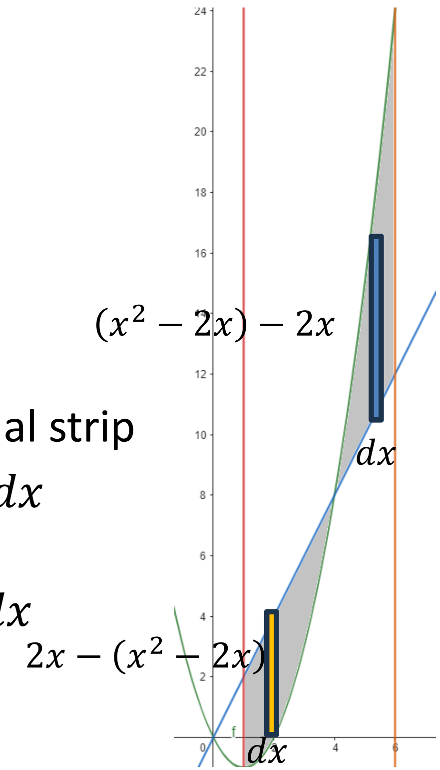
   $x(x-4) = 0 \Rightarrow x = 0,4$

   $[1,4], [4,6]$

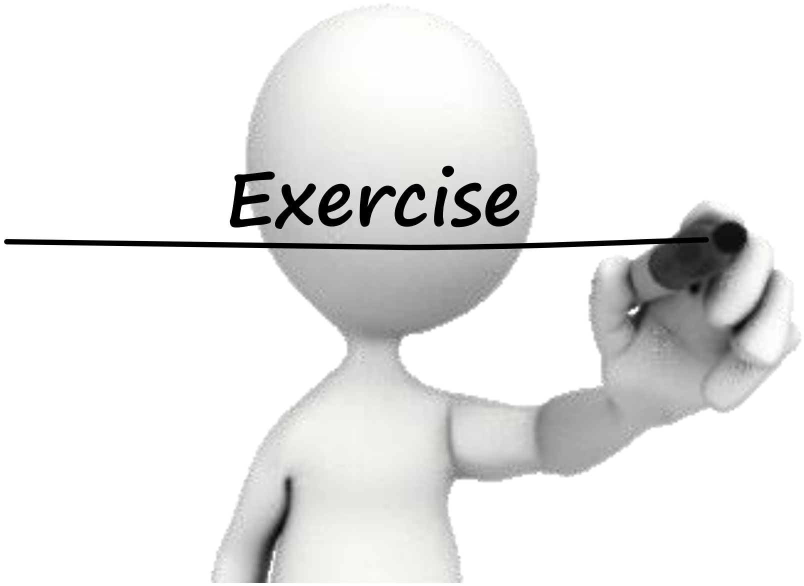
5. Integrate the area of infinitesimal strips

   $\int_4^6 |x^2 - 4x| dx$

   $= \int_1^4 (4x - x^2)dx + \int_4^6 (x^2 - 4x)dx$



Area between curves = total area


Exercise

Problems

$$\int \frac{\sin x}{(1+\cos x)^3}\,dx =$$

(a) $\dfrac{1}{2(1+\cos x)^2} + C$

(b) $\dfrac{1}{(1+\cos x)^2} + C$

(c) $\dfrac{-1}{2(1+\cos x)^2} + C$

(d) $\dfrac{1}{4(1+\cos x)^4} + C$

(e) $\dfrac{-1}{4(1+\cos x)^4} + C$

Problems

Which of the following integrals gives the area of the region bounded by the curves $x = y^2$ and $x = 6 - y$?

(a) $\int_{-3}^{2} (6 - y - y^2)\, dy$

(b) $\int_{-3}^{2} (y^2 - 6 + y)\, dy$

(c) $\int_{4}^{9} (6 - x - \sqrt{x})\, dy$

(d) $\int_{4}^{9} (\sqrt{x} - 6 + x)\, dy$

(e) $\int_{4}^{9} (6 - y - y^2)\, dy$

Problems

Evaluate $\displaystyle\int_{\sqrt{2}}^{2} \frac{4x}{x^2-1}\,dx$

(a) $2\ln 3$

(b) $2\ln 2 - 2\ln\sqrt{2}$

(c) $\ln 3 - 1$

(d) $2\ln 3 - 2$

(e) $2\ln 2$

Problems

Consider the region R bounded by $y = x^2$, $y = \sqrt{x}$ on the interval from $x = 0$ to $x = 2$. Which of the following gives the area of R?

(a) $\int_0^1 (\sqrt{x} - x^2)\, dx + \int_1^2 (x^2 - \sqrt{x})\, dx$

(b) $\int_0^1 (x^2 - \sqrt{x})\, dx + \int_1^2 (\sqrt{x} - x^2)\, dx$

(c) $\int_0^2 (\sqrt{x} - x^2)\, dx$

(d) $\int_0^2 (x^2 - \sqrt{x})\, dx$

(e) None of these

Problems

Evaluate $\displaystyle\int_{\pi^2/16}^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

(a) $\dfrac{\sqrt{2}}{2}+1$

(b) $2+\sqrt{2}$

(c) $\dfrac{\sqrt{2}}{2}-1$

(d) $-2+\sqrt{2}$

(e) $1-\sqrt{2}$

Rewrite :

Id f and g'

- $f =$
- $g =$
- $g' =$

u-sub

complete substitution
for the limits

Evaluate the integral:

Problems

Find the area bounded by $y + x^2 = 6$ and $y + 2x - 3 = 0$.

(a) $\dfrac{16}{3}$

(b) $\dfrac{40}{3}$

(c) $\dfrac{20}{3}$

(d) $\dfrac{32}{3}$

(e) None of the above

Problems

Find the area of the region bounded by $x = y^2$ and $x = y + 2$.

(a) $\frac{9}{2}$

(b) $\frac{3}{2}$

(c) $\frac{19}{6}$

(d) $\frac{16}{3}$

(e) None of the above

Problems

Calculate the area of the region bounded by the curves $4x + y^2 = 12$ and $x = y$.

(a) 21

(b) $\frac{62}{3}$

(c) $\frac{64}{3}$

(d) 22

(e) $\frac{73}{3}$