Math 308: Week-in-Review 6 (EXAM 1 REVIEW)

Characterizing Differential Equations

Review

- The order of a differential equation is the order of the highest derivative.
- Ordinary vs Partial Differential Equations:
  - An ordinary differential equation has derivatives with respect to one variable.
  - A partial differential equation has derivatives with respect to more than one variable.
- Linear ODEs:
  - A linear ODE has the form

$$a_n(x)y^{(n)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x).$$

  - Conditions:
    - All the y's are in different terms.
    - None of the y's are inside a function or to a power.
    - The y's can be multiplied by a function of x.
    - There can be terms that depend only on x.
- Homogeneous Linear ODEs:
  - A linear ODE is homogeneous if the $g(x)$ term is 0.
- Separable ODEs:
  - An ODE is separable if you can write it in the form $y' = f(x)g(y)$.
- Autonomous ODEs:
  - An ODE is autonomous if the independent variable (x) does not show up explicitly, i.e., if x does not show up outside of y.

1. Classify the following differential equations into one (or more) of the following categories and state the order: Partial differential equation, Ordinary differential equation, Separable, Linear, Homogeneous, Autonomous.

   (a) $y^2 - y'' + 6 = 0$

   (b) $f_x - f_y = xf$

   (c) $y'(x) + x^2 y(x) = 3y(x)$

   (d) $g' = x^2 \sin(g)$

   (e) $\sin(x)w''' + w - 3 = 0$

   (f) $u''(x) = \sin(u(x))$

   (g) $f^{(5)} - \cos(x^2)f''' - \tan(x)f = 3\tan(x)$

Solving Differential Equations

Review

- First Order ODEs:
  - You do NOT need to guess which method to use to solve a 1st order ODE!
  - How to determine which method to use:
    (a) Is the equation separable? If yes, use separation of variables.
    (b) Is the equation linear? If yes, use the method of integrating factors.
    (c) Is it a Bernoulli equation? If yes, then use $v = y^{1-n}$.
    (d) Is the equation exact? If yes, then use the method for exact equations.
    (e) Is it a homogeneous equation? If yes, then use $v = y/x$ to get a separable equation.
    (f) If none of the above, then try to find an integrating factor to make the equation exact.
- Second Order Linear ODEs:
  - Homogeneous with constant coefficients:
    (a) Look for solutions of the form $y(t) = e^{rt}$.
    (b) Find the characteristic equation.
    (c) Find the roots of the characteristic equation.
    (d) The general solution is given by:
      * Distinct real roots: $c_1e^{r_1t} + c_2e^{r_2t}$
      * Complex roots: $c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt)$
      * Repeated real roots: $c_1e^{rt} + c_2te^{rt}$
    (e) If you have initial conditions, use them to solve for c_1 and c_2.
  - Nonhomogeneous:
    * Method of undetermined coefficients (if constant coefficients and you can guess).
    * Variation of parameters.

2. Find the general solution to

$$t^2y' + ty - t = 0.$$

3. Solve the initial value problem

$$u' - tu^{-2} = 0, \quad u(1) = -1.$$

4. Find the general solution to

$$f'' = 3f' - 2f.$$

5. Find the general solution to

$$w'' + 4w' + 4w = 5e^t.$$

6. Find the general solution to

$$(4x - 2y)y' + 4y = -2x.$$

7. Find the general solution to

$$3g'' - 2g' + 4 = 0.$$

8. Solve the initial value problem

$$f = -\frac{1}{9}f'', \quad f(0) = -2, \quad f'(0) = 1.$$

9. Suppose you wanted to use the method of undetermined coefficients to find a particular solution to

$$y'' - 5y' + 6y = 4e^{-2t} + 3t^3.$$

What is an appropriate guess for the particular solution y_p?

10. Suppose you wanted to use the method of undetermined coefficients to find a particular solution to

$$y'' - 2y' + y = 3te^{t} - t\sin(t).$$

What is an appropriate guess for the particular solution y_p?

11. Given that x^2 and x^{-1} are solutions to the corresponding homogeneous equation, find a particular solution to

$$x^2y'' - 2y = 3x^2 - 1, \quad x > 0.$$

Analysis of ODEs

Review

- Where is a solution valid?
  - Solution is valid on a single interval where the solution is a function that is defined and differentiable.
- Existence and Uniqueness:
  - 1st order linear ODEs: If p and g are continuous on an interval $I = (a, b)$ containing the initial condition t_0, then the initial value problem

    $$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

    has a unique solution on I.
  - 1st order nonlinear ODEs: Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $(a, b) \times (c, d)$ containing the point (t_0, y_0). Then, there is a unique solution to the initial value problem

    $$y' = f(t, y), \quad y(t_0) = y_0$$

    on a sufficiently small interval $I_h = (t_0 - h, t_0 + h)$ around t_0.
  - 2nd order linear ODEs: Consider the initial value problem

    $$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0.$$

    If p, q, and g are continuous on an open interval $I = (a, b)$ that contains the point t_0, then there is exactly one solution to the initial value problem and the solution exists throughout the entire interval I.
- The Wronskian of y_1 and y_2 is defined by

  $$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{vmatrix} = y_1(t)y'_2(t) - y'_1(t)y_2(t).$$

- $\{y_1, y_2\}$ is a fundamental set of solutions means that the general solution is $c_1y_1 + c_2y_2$.
- Slope fields
- Equilibrium solutions
- Stability of equilibrium solutions:
  - (Asymptotically) stable: If you start near it, you go in towards it.
  - Unstable: If you start near it, you go away from it.
  - Semistable: If you start near on one side, you go towards it, but if you start near on the other side, you go away from it.
- Phase line diagrams

12. Without solving the initial value problem, where is a unique solution guaranteed to exist?

$$y' - t^2 \tan(t) y = \sqrt{4-t}, \quad y(0) = \pi.$$

13. Without solving the initial value problem, where is a unique solution guaranteed to exist?

$$(t-1)w'' + w' - \ln(t+3)w = t^3 \cos(t), \quad w(2) = -2, \quad w'(2) = 7.$$

14. For which values t_0 and y_0 is the following initial value problem guaranteed to have a unique solution?

$$t^2 y^2 - (t+y)y' = 0, \quad y(t_0) = y_0.$$

15. Show that x and xe^x form a fundamental set of solutions to

$$x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0.$$

16. Solve for the explicit solution $u(x)$. Where is the solution to the initial value problem valid? How does this depend on a?

$$u' = u^2, \quad u(0) = a.$$

17. Consider the differential equation

$$f' = f(f-2)^2(f-4).$$

(a) Find the equilibrium solutions.

(b) Draw the phase line diagram.

(c) Sketch the slope field.

(d) Determine the stability of each equilibrium solution.

(e) Determine $\lim_{t\to\infty} f(t)$ for different initial values $f(0)$.