Math 151 - Week-In-Review 6

Topics for the week:

3.3 Derivatives of Trigonometric Functions
3.4 The Chain Rule
3.5 Implicit Differentiation

3.3 Derivatives of Trigonometric Functions

1. Compute the derivative of each of the six trigonometric functions.

$\frac{d}{dx}[\sin(x)] = \cos(x)$

$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$

$\frac{d}{dx}[\cos(x)] = -\sin(x)$

$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$

$\frac{d}{dx}[\tan(x)] = \sec^2(x)$

$\frac{d}{dx}[\cot(x)] = \csc^2(x)$

2. Compute $\frac{dg(w)}{dw}$ for $g(w) = \frac{4w^2+5w}{\sin(w)}$.

$$g(w) = \frac{4w^2+5w}{\sin(w)}$$

$$\frac{dg(w)}{dw} = \frac{(8w+5)\sin(w) - \cos(w)(4w^2+5w)}{(\sin(w))^2}$$

$$\frac{dg(w)}{dw} = \frac{8w\sin(w) + 5\sin(w) - 4w^2\cos(w) - 5w\cos(w)}{\sin^2(w)}$$

3. For $y = e^x\tan(x)$, find $\frac{dy}{dx}$.

$$y = e^x\tan(x)$$

$$\frac{dy}{dx} = e^x\tan(x) + \sec^2(x)e^x$$

$$\frac{dy}{dx} = e^x(\tan(x) + \sec^2(x))$$

or

$$\frac{dy}{dx} = e^x(\tan(x) + \tan^2(x) + 1)$$

4. Compute $\frac{d^{101}}{dx^{101}}[-11\cos(x)]$

$\frac{d}{dx}\left[-11\cos(x)\right] = +11\sin(x)$

$\frac{d^2}{dx^2}\left[-11\cos(x)\right] = 11\cos(x)$

$\frac{d^3}{dx^3}\left[-11\cos(x)\right] = -11\sin(x)$

$\frac{d^4}{dx^4}\left[-11\cos(x)\right] = -11\cos(x)$

so $\frac{101}{4} = 25\frac{1}{4}$

$\frac{d^{101}}{dx^{101}}\left[-11\cos(x)\right] = 11\sin(x)$

5. Determine $f'(\theta)$ for $f(\theta) = -7\sqrt{\theta}\cdot\tan(\theta)$

$f(\theta) = -7\theta^{1/2}\cdot\tan(\theta)$

$f'(\theta) = -\frac{7}{2}\theta^{-1/2}\cdot\tan(\theta) + \sec^2(\theta)\cdot(-7\theta^{1/2})$

$f'(\theta) = -\frac{7\tan(\theta)}{2\sqrt{\theta}} - 7\sqrt{\theta}\sec^2(\theta)$

6. Determine the points on the curve $y = 3\sec(x) - 3$ that have a horizontal tangent for $0 \le x < 2\pi$.

$y = 3\sec(x) - 3$ $\qquad x \neq \frac{\pi}{2}$ or $\frac{3\pi}{2}$ in $0 \le x < 2\pi$

$\frac{dy}{dx} = 3\sec(x)\tan(x)$

horizontal tangent means $m_{tan} = 0$ or $\frac{dy}{dx} = 0$ $\rightarrow$ $0 = 3\sec(x)\tan(x)$

$3\sec(x) = 0$ Never $\qquad$ or $\qquad \tan(x) = 0$

$x = 0 + \pi k$

$\boxed{x = 0 \quad x = \pi}$

3.4 The Chain Rule

7. For $g(x) = \cot^2(x)$ compute the first and second derivatives of $g(x)$.

$g(x) = \cot^2(x) = (\cot(x))^2$

$g'(x) = 2(\cot(x))(-\csc^2(x))$

$g'(x) = -2\cot(x)\csc^2(x)$

$g''(x) = -2(-\csc^2(x))\cdot\csc^2(x) + 2(\csc(x))(-\csc(x)\cot(x))(-2\cot(x))$

$g''(x) = 2\csc^4(x) + 4\csc^2(x)\cot^2(x)$

$= 2\csc^2(x)(\csc^2(x) + 2\cot^2(x))$

8. Differentiate $g(t) = 7(2t^{-2} - 5t^4)^2$ with respect to t.

$g(t) = 7(2t^{-2} - 5t^4)^2$

$g'(t) = 7\cdot 2(2t^{-2} - 5t^4)^1(-4t^{-3} - 20t^3)$

$g'(t) = 14(2t^{-2} - 5t^4)(-4t^{-3} - 20t^3)$

9. Compute the derivative of $h(x) = \sqrt[3]{x^3 + 3x^2 - 6x - 8}$ with respect to x.

$h(x) = (x^3 + 3x^2 - 6x - 8)^{1/3}$

$\frac{dh(x)}{dx} = \frac{1}{3}(x^3 + 3x^2 - 6x - 8)^{-2/3} \cdot (3x^2 + 6x - 6)$

$\frac{dh(x)}{dx} = \frac{x^2 + 2x - 2}{\sqrt[3]{(x^3 + 3x^2 - 6x - 8)^2}}$

10. Compute the derivative of $s(t) = 6(5t^2 - 4t)^8 + \arcsin(3t)$ with respect to t.

$s(t) = 6(5t^2 - 4t)^8 + \arcsin(3t)$

$s'(t) = 48(5t^2 - 4t)^7(10t - 4) + \frac{1}{\sqrt{1-(3t)^2}} \cdot 3$

$s'(t) = 48(5t^2 - 4t)^7(10t - 4) + \frac{3}{\sqrt{1-9t^2}}$

11. Find the 28th derivative of $f(x) = e^{4x} + x^3$.

$f(x) = e^{4x} + x^3$

$\frac{df(x)}{dx} = 4e^{4x} + 3x^2$

$\frac{d^2f(x)}{dx^2} = \underbrace{4 \cdot 4}_{4^2}e^{4x} + \underbrace{6x}$ derivative $\rightarrow$ goes to 0 at 4th derivative

$\frac{d^{28}f(x)}{dx^{28}} = 4^{28}e^{4x}$

12. Write the equation of the line tangent to $h(x) = \left(\frac{2x^2}{x+1}\right)^4$ at $x = 1$.

$h(x) = \left(\frac{2x^2}{x+1}\right)^4$

$h(1) = \left(\frac{2}{2}\right)^4 = 1$

$h'(x) = 4\left(\frac{2x^2}{x+1}\right)^3\left(\frac{4x(x+1) - 1(2x^2)}{(x+1)^2}\right)$

$h'(1) = 4\left(\frac{2}{2}\right)^3\left(\frac{4(2) - 2}{(2)^2}\right)$

$= 4(1)^3\left(\frac{6}{4}\right)$

$= 6$

$y = f'(a)(x-a) + f(a)$

$y = 6(x-1) + 1$

or

$y - y_1 = m(x - x_1)$

$y - 1 = 6(x-1)$

13. Differentiate $f(r) = 4^{2r^2 - r^3}$

$f(r) = 4^{2r^2 - r^3}$

$f'(r) = 4^{2r^2 - r^3} \ln(4) \cdot (4r - 3r^2)$

14. Given $y = \dfrac{x^3 - 11}{1 - x^2}$ and $\dfrac{dy}{dx} = \dfrac{-x^4 + 3x^2 - 22x}{(1 - x^2)^2}$, compute second derivative of y with respect to x.

$y = \dfrac{x^3 - 11}{1 - x^2}$ $\qquad \dfrac{dy}{dx} = \dfrac{-x^4 + 3x^2 - 22x}{(1 - x^2)^2}$

$\dfrac{d^2y}{dx^2} = \dfrac{(-4x^3 + 6x - 22)(1 - x^2)^2 - 2(1 - x^2)(-2x)(-x^4 + 3x^2 - 22x)}{(1 - x^2)^4}$

$\dfrac{d^2y}{dx^2} = \dfrac{(1 - x^2)\left[-4x^3 + 6x - 22 + 4x^5 - 6x^3 + 22x^2 - 4x^5 + 12x^3 - 88x^2\right]}{(1 - x^2)^4}$

$\dfrac{d^2y}{dx^2} = \dfrac{2x^3 - 66x^2 - 22}{(1 - x^2)^3}$

15. For $f(\theta) = \theta \cos(\theta^2)$ compute $f'(\theta)$ and $f''(\theta)$.

$f(\theta) = \theta \cdot \cos(\theta^2)$

$f'(\theta) = 1 \cos(\theta^2) + (-\sin(\theta^2) \cdot 2\theta)\theta$

$f'(\theta) = \cos(\theta^2) - 2\theta^2 \sin(\theta^2)$

$f''(\theta) = -\sin(\theta^2) \cdot 2\theta - 4\theta \sin(\theta^2) + \cos(\theta^2) \cdot 2\theta(-2\theta^2)$

$f''(\theta) = -6\theta \sin(\theta^2) - 4\theta^3 \cos(\theta^2)$

3.5 Implicit Differentiation

16. Compute $\frac{dy}{dx}$ for $\tan(x^2 + y^2) = \sin(y)$.

$$\tan(x^2+y^2) = \sin(y)$$

$$\sec^2(x^2+y^2)\left(2x + 2y\frac{dy}{dx}\right) = \cos(y)\frac{dy}{dx}$$

$$2x\sec^2(x^2+y^2) + 2y\sec^2(x^2+y^2)\frac{dy}{dx} = \cos(y)\frac{dy}{dx}$$

seperate $\frac{dy}{dx}$ terms

$$2x\sec^2(x^2+y^2) = \cos(y)\frac{dy}{dx} - 2y\sec^2(x^2+y^2)\frac{dy}{dx}$$

$$2x\sec^2(x^2+y^2) = \frac{dy}{dx}\left[\cos(y) - 2y\sec^2(x^2+y^2)\right]$$

$$\frac{2x\sec^2(x^2+y^2)}{\cos(y) - 2y\sec^2(x^2+y^2)} = \frac{dy}{dx}$$

17. Compute $\frac{dy}{dx}$ for $\sin(2y)e^{x^2} = \cos(x^3 + y^2)$.

$$\sin(2y)e^{x^2} = \cos(x^3+y^2)$$

$$\cos(2y)\,2\frac{dy}{dx}e^{x^2} + e^{x^2}\cdot 2x\cdot\sin(2y) = -\sin(x^3+y^2)\left(3x^2 + 2y\frac{dy}{dx}\right)$$

$$2\cos(2y)e^{x^2}\cdot\frac{dy}{dx} + 2x\sin(2y)e^{x^2} = -3x^2\sin(x^3+y^2) - 2y\sin(x^3+y^2)\frac{dy}{dx}$$

$$2\cos(2y)e^{x^2}\frac{dy}{dx} + 2y\sin(x^3+y^2)\frac{dy}{dx} = -3x^2\sin(x^3+y^2) - 2x\sin(2y)e^{x^2}$$

$$\frac{dy}{dx}\left[2\cos(2y)e^{x^2} + 2y\sin(x^3+y^2)\right] = -3x^2\sin(x^3+y^2) - 2x\sin(2y)e^{x^2}$$

$$\frac{dy}{dx} = \frac{-3x^2\sin(x^3+y^2) - 2x\sin(2y)e^{x^2}}{2\cos(2y)e^{x^2} + 2y\sin(x^3+y^2)}$$

18. Determine the tangent and normal to the curve $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$.

$$x^2 - xy + y^2 = 7$$

$$2x - 1y + \frac{dy}{dx}(-x) + 2(y)'\frac{dy}{dx} = 0$$

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$-x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x + y$$

$$\frac{dy}{dx}\left[-x + 2y\right] = -2x + y$$

$$\frac{dy}{dx} = \frac{-2x + y}{-x + 2y}$$

$$m_{tan} = \frac{-2(-1) + (2)}{-(-1) + 2(2)}$$

$$= \frac{4}{5}$$

$$m_{norm} = -5/4$$

Tangent Line

$$y = \frac{4}{5}(x + 1) + 2$$

Normal Line

$$y = -\frac{5}{4}(x + 1) + 2$$

19. Compute $\frac{dy}{dx}$ for $5^y + x^2y^3 = 1 + y\cos(x^2)$

$$5^y + x^2y^3 = 1 + y\cos(x^2)$$

$$5^y \ln(5) \cdot \frac{dy}{dx} + 2xy^3 + 3y^2\frac{dy}{dx} \cdot x^2 = \frac{dy}{dx}\cos(x^2) - \sin(x^2)\,2x \cdot y$$

$$5^y \ln(5)\frac{dy}{dx} + 3x^2y^2\frac{dy}{dx} - \cos(x^2)\frac{dy}{dx} = -2xy\sin(x^2) - 2xy^3$$

$$\frac{dy}{dx}\left[5^y \ln(5) + 3x^2y^2 - \cos(x^2)\right] = -2xy\sin(x^2) - 2xy^3$$

$$\frac{dy}{dx} = \frac{-2xy\sin(x^2) - 2xy^3}{5^y\ln(5) + 3x^2y^2 - \cos(x^2)}$$