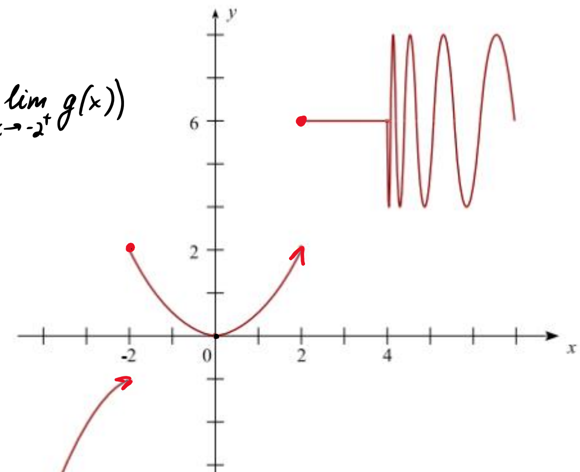
Math 151/171 **WEEK in REVIEW 3** **Spring 2024**

1. For the function g whose graph is given, state the value of the given quantity, if it exists.



$\lim_{x\to 2} g(x)$ DNE

(a) $\lim_{x\to -2^-} g(x)$ $= -1$

(b) $\lim_{x\to -2^+} g(x)$ $= 2$

(c) $\lim_{x\to -2} g(x)$ DNE $\left(\lim_{x\to -2^-} g(x) \neq \lim_{x\to -2^+} g(x)\right)$

(d) $g(-2)$ $= 2$

(e) $\lim_{x\to 0} g(x)$ $= 0$

(f) $g(0)$ $= 0$

(g) $\lim_{x\to 2^-} g(x)$ $= 2$

(h) $\lim_{x\to 2^+} g(x)$ $= 6$

(i) $g(2)$ $= 6$

(j) $\lim_{x\to 4^-} g(x)$ $= 6$

(k) $\lim_{x\to 4^+} g(x)$ DNE (the values are oscilating between 4 and 8, and do not approach any number)

(l) $\lim_{x\to 4} g(x)$ DNE

2. Find all holes and vertical asymptote(s) for the graph of

$$g(x) = \frac{(x^2+5x)(x-2)}{(x+1)(x^2+4x-5)} = \frac{x\cancel{(x+5)}(x-2)}{(x+1)\cancel{(x+5)}(x-1)} = \frac{x(x-2)}{(x+1)(x-1)}$$

and determine the behavior of the function near the vertical asymptotes.

holes: $x+5=0$ or $\boxed{x=-5}$ a hole

Vertical asymptotes:

$x+1=0$ or $\boxed{x=-1}$

$x-1=0$ or $\boxed{x=1}$

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behavior of the function near $x=-1$: $g(-1.1) = \frac{-1.1(-1.1-2)}{(-1.1+1)(-1.1-1)} = \infty$

$g(-0.9) = \frac{-0.9(-2-0.9)}{(1-0.9)(-0.9-1)} = -\infty$

near $x=1$: $g(1.1) = \frac{1.1(1.1-2)}{(1.1-1)(1.1+1)} = -\infty$

$g(0.9) = \frac{0.9(0.9-2)}{(0.9+1)(0.9-1)}$ $+\infty$

$\lim_{x\to -1^-} g(x) = \infty$ $\qquad \lim_{x\to 1^-} g(x) = \infty$

$\lim_{x\to -1^+} g(x) = -\infty$ $\qquad \lim_{x\to 1^+} g(x) = -\infty$

3. Calculate the following limits or state why the limit does not exist. Do not use the L'Hospital's Rule.

(a) $\lim_{x \to 4} \sqrt{x + \sqrt{x}}$ $= \sqrt{4+\sqrt{4}} = \sqrt{6}$

(b) $\lim_{x \to 5} \frac{5x - x^2}{x^2 - 4x - 5}$ $= \frac{0}{0} = \lim_{x \to 5} \frac{-x\cancel{(-5+x)}}{\cancel{(x-5)}(x+1)} = -\lim_{x \to 5} \frac{x}{x+1} = -\frac{5}{6}$

(c) $\lim_{h\to 0} \dfrac{(3+h)^{-1} - 3^{-1}}{h}$ $= \dfrac{0}{0} = \lim_{h\to 0} \dfrac{\frac{1}{3+h} - \frac{1}{3}}{h} = \lim_{h\to 0} \dfrac{\frac{3-(3+h)}{3(3+h)}}{h}$

$= \lim_{h\to 0} \dfrac{3-(3+h)}{3h(3+h)} = \lim_{h\to 0} \dfrac{3-3-h}{3h(3+h)} = \lim_{h\to 0} \dfrac{-\cancel{h}}{3\cancel{h}(3+h)} = -\dfrac{1}{3(3+0)} = -\dfrac{1}{9}$

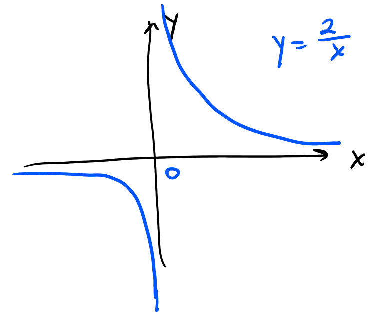
$(a-b)(a+b) = a^2 - b^2$

(d) $\lim_{x \to 3} \frac{x - \sqrt{4x-3}}{x^2 - 9} = \frac{0}{0} = \lim_{x \to 3} \frac{(x - \sqrt{4x-3})(x + \sqrt{4x-3})}{(x^2-9)(x+\sqrt{4x-3})} = \lim_{x \to 3} \frac{x^2 - (\sqrt{4x-3})^2}{(x^2-9)(x+\sqrt{4x-3})}$

$$= \lim_{x \to 3} \frac{x^2 - 4x + 3}{(x-3)(x+3)(x+\sqrt{4x-3})} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x+3)(x+\sqrt{4x-3})}$$

$$= \lim_{x \to 3} \frac{x-1}{(x+3)(x+\sqrt{4x-3})} = \frac{2}{6(3+3)} = \boxed{\frac{1}{18}}$$

(e) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{|x|}\right)$ DNE



$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$|x| = x$, $\lim_{\substack{x\to 0^+ \\ x>0}}\left(\frac{1}{x}-\frac{1}{x}\right) = 0$

$|x| = -x$, $\lim_{\substack{x\to 0^- \\ x<0}}\left(\frac{1}{x}-\frac{1}{-x}\right) = \lim_{x\to 0^-}\left(\frac{2}{x}\right) = -\infty$

$5 \neq -5$

(f) $\lim_{x\to 2} \frac{x^2+x-6}{|x-2|} = \lim_{x\to 2} \frac{(x+3)(x-2)}{|x-2|}$ $\boxed{DNE}$

$$|x-2| = \begin{cases} x-2, & \text{if } x-2>0 \\ -(x-2), & \text{if } x-2<0 \end{cases} \quad \text{or} \quad |x-2| = \begin{cases} x-2, & \text{if } x>2 \\ 2-x, & \text{if } x<2 \end{cases}$$

$$\lim_{x\to 2^+} \frac{(x+3)(x-2)}{|x-2|} \quad \frac{x>2}{|x-2| = x-2} \quad \lim_{x\to 2^+} \frac{(x+3)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x\to 2^+}(x+3) = 5$$

$$\lim_{x\to 2^-} \frac{(x+3)(x-2)}{|x-2|} \quad \frac{x<2}{|x-2| = 2-x} \quad \lim_{x\to 2^-} \frac{(x+3)\cancel{(x-2)}}{-1\cancel{(2+x)}} = -\lim_{x\to 2^-}(x+3) = -5$$

(g) $\lim_{t\to 5}\left\langle \frac{2t-10}{t-5}, \frac{5-t}{t^2-4t-5}\right\rangle = \boxed{\langle 2, -\frac{1}{6}\rangle}$

$$\lim_{t\to 5}\frac{2t-10}{t-5} = \frac{0}{0} = \lim_{t\to 5}\frac{2\cancel{(t-5)}}{\cancel{t-5}} = \boxed{2}$$

$$\lim_{t\to 5}\frac{5-t}{t^2-4t-5} = \lim_{t\to 5}\frac{-\cancel{(t-5)}}{\cancel{(t-5)}(t+1)} = -\lim_{t\to 5}\frac{1}{t+1} = \boxed{-\frac{1}{6}}$$

(h) $\lim_{x \to 0} x^4 \cos \frac{1}{x^2}$

Use the Squeeze Thm.

If $g(x) < f(x) < h(x)$ for all x, and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$

$-1(x^4) \le x^4 \cos \frac{1}{x^2} \le 1(x^4)$

$-x^4 \le x^4 \cos \frac{1}{x^2} \le x^4 \implies$ By the Squeeze Thm, $\boxed{\lim_{x \to 0} x^4 \cos \frac{1}{x^2} = 0}$

$\lim_{x \to 0} (-x^4) = 0$

$\lim_{x \to 0} x^4 = 0$

4. If $2x - 2 \leq f(x) \leq x^2 - 2x + 2$ for $x \geq 0$, find $\lim_{x \to 2} f(x)$.

Use the Squeeze Thm.

$$\underbrace{(2x-2)}_{2,\text{ as } x \to 2} \leq f(x) \leq \underbrace{(x^2-2x+2)}_{2 \text{ as } x \to 2} \Longrightarrow \boxed{\lim_{x \to 2} f(x) = 2}$$

$$\lim_{x \to 2} (2x-2) = 4-2 = 2$$

$$\lim_{x \to 2} (x^2 - 2x+2) = 4-4+2 = 2$$

5. Let

$$f(x) = \begin{cases} \sqrt{-x}, & \text{if } x < 0 \\ 3 - x, & \text{if } 0 \le x < 3 \\ (x-3)^2, & \text{if } x > 3 \end{cases}$$

Evaluate each limit if exists.

(a) $\lim_{x \to 0} f(x)$

(b) $\lim_{x \to 3} f(x)$

(a) $\lim_{x \to 0^-} f(x) \; \frac{x<0}{f(x) = \sqrt{-x}} \; \lim_{x \to 0} \sqrt{-x} = 0$

$\lim_{x \to 0^+} f(x) \; \frac{x>0}{f(x) = 3-x} \; \lim_{x \to 0} (3-x) = 3$

$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\boxed{\lim_{x \to 0} f(x) \text{ DNE}}$

(b) $\lim_{x \to 3^-} f(x) \; \frac{x<3}{f(x) = 3-x} \; \lim_{x \to 3} (3-x) = 0$

$\lim_{x \to 3^+} f(x) \; \frac{x>3}{f(x) = (x-3)^2} \; \lim_{x \to 3} (x-3)^2 = 0$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 0$

$\boxed{\lim_{x \to 3} f(x) = 0}$

$f(x)$ is continuous @ a if $\lim_{x\to a} f(x) = f(a) \Rightarrow \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

6. Find the x-value at which f is discontinuous and determine whether f is continuous from the right, or from the left, or neither.

$$f(x) = \begin{cases} 1+x^2, & \text{if } x \le 0 \\ 4-x, & \text{if } 0 < x \le 4 \\ (x-4)^2, & \text{if } x > 4 \end{cases}$$

continuous from the right, when $\boxed{\lim_{x\to a^+} f(x) = f(a)}$

continuous from the left when $\boxed{\lim_{x\to a^-} f(x) = f(a)}$

$x=0$: $f(0) = 1+0^2 = 1$

$\lim_{x\to 0^-} f(x) \xrightarrow[f(x)=1+x^2]{x<0} \lim_{x\to 0} (1+x^2) = 1$

$\lim_{x\to 0^+} f(x) \xrightarrow[f(x)=4-x]{x>0} \lim_{x\to 0} (4-x) = 4$

$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x) \Rightarrow f(x)$ has a jump discontinuity @ $x=0$.

$\lim_{x\to 0^-} f(x) = 1 = f(0)$

$f(x)$ is continuous from the left @ $x=0$.

$x=4$. $f(4) = 4-4 = 0$

$\lim_{x\to 4^-} f(x) \xrightarrow[f(x)=4-x]{x<4} \lim_{x\to 4} (4-x) = 0$

$\lim_{x\to 4^+} f(x) \xrightarrow[f(x)=(x-4)^2]{x>4} \lim_{x\to 4} (x-4)^2 = 0$

$f(x)$ is continuous @ $x=4$.

7. Find the value(s) of x where the function $f(x)$ is discontinuous. If the discontinuity, $x = a$, is removable, find a function g that agrees with f for all values of x and is continuous at $x = a$.

(a) $f(x) = \dfrac{x-4}{x^2+x-20} = \dfrac{\cancel{x-4}}{(x+5)\cancel{(x-4)}} = \dfrac{1}{x+5}$

$x-4 \overset{x=4}{=} 0$ is a removable discontinuity.

$f(x) = \dfrac{1}{x+5} \Rightarrow f(4) = \dfrac{1}{9}$

$$\boxed{g(x) = \begin{cases} \dfrac{x-4}{x^2+x-20}, & \text{if } x \neq 4 \\ \dfrac{1}{9}, & \text{if } x = 4 \end{cases}}$$

$g(x)$ is continuous @ $x = 4$.

$f(x) = \dfrac{x-4}{(x+5)(x-4)}$

if $x+5=0$ or $x=-5$

f has an infinite discontinuity @ $x=-5$

(b) $f(x) = \dfrac{x^2-2x-8}{x^2-x-6} = \dfrac{(x-4)\cancel{(x+2)}}{(x-3)\cancel{(x+2)}} = \dfrac{x-4}{x-3}$

$x=3$ and $x=-2$

$f(x)$ has a removable discontinuity at $x=-2$

$h(x) = \dfrac{x-4}{x-3}$, $h(-2) = \dfrac{-2-4}{-2-3} = \dfrac{6}{5}$

$$\boxed{g(x) = \begin{cases} \dfrac{x^2-2x-8}{x^2-x-6}, & \text{if } x \neq -2 \\ \dfrac{6}{5}, & \text{if } x = -2 \end{cases}}$$

$g(x)$ is continuous @ $x=-2$

if $x=3$, $f(x)$ has an infinite discontinuity.

$a^2-b^2=(a-b)(a+b)$

8. Find the values of a and b that make f continuous everywhere.

$$f(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & \text{if } x<2\\ ax^2-bx+3, & \text{if } 2\le x<3\\ 4xa+b, & \text{if } x\ge 3\end{cases}$$

$\underline{x=2}$

$\lim_{x\to 2^-} f(x)$ $\frac{x<2}{f(x)=\frac{x^2-4}{x-2}}$ $\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}=\lim_{x\to 2}(x+2)=4$

$\lim_{x\to 2^+} f(x)$ $\frac{x>2}{f(x)=ax^2-bx+3}$ $\lim_{x\to 2}(ax^2-bx+3)=4a-2b+3$

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x) \Rightarrow \boxed{4a-2b+3=4}$ or $\boxed{4a-2b=1}$

$f(2)=a(2^2)-b(2)+3=4a-2b+3$

$\underline{x=3}$ $f(3)=4(3)a+b=12a+b$

$\lim_{x\to 3^-} f(x)$ $\frac{x<3}{f(x)=ax^2-bx+3}$ $\lim_{x\to 3}(ax^2-bx+3)=3^2a-3b+3=9a-3b+3$

$\lim_{x\to 3^+} f(x)$ $\frac{x>3}{f(x)=4ax+b}$ $\lim_{x\to 3}(4ax+b)=12a+b$

$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x) \Rightarrow 9a-3b+3=12a+b$

$12a-9a+b+3b=3$ or $\boxed{3a+4b=3}$

The system for a and b

$\begin{cases}4a-2b=1\\ 3a+4b=3\end{cases} \Rightarrow 2b=4a-1$, plug into the second equation

$3a+2(2b)=3$

$3a+2(4a-1)=3$

$3a+8a-2=3$

$11a=5$ or $\boxed{a=\frac{5}{11}}$

$2b=4a-1$ or $b=\frac{1}{2}(4a-1)$

$2b=4\cdot\frac{5}{11}-1$ $b=\frac{1}{2}\left(\frac{20}{11}-1\right)=\frac{1}{2}\left(\frac{20-11}{11}\right)=\frac{9}{22}$

$2b=\frac{20}{11}-1$

$2b=\frac{9}{11} \Rightarrow \boxed{b=\frac{9}{22}}$

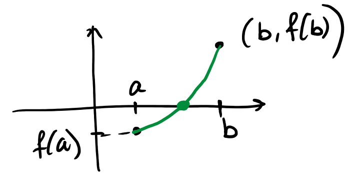
9. Use the Intermediate Value Theorem to show that there is a root of the equation $x^4 + x = 3$ in the interval $(1, 2)$.

$f(x) = x^4 + x - 3$ interval $(1,2)$

$f(1) = 1 + 1 - 3 = -1$

$f(2) = 2^4 + 2 - 3 = 15$

$f(1) < 0$ and $f(2) = 15$.



There is a point a in $(1,2)$ such ~~that~~ that $f(a) = 0$.

a is a root of the equation $x^4 + x = 3$.