Math 151
Week-In-Review 8

Exam 2 Review
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Problem Statements

1. Find the derivative of the following functions.

$\frac{1}{5} \cdot x^{-1/5}$

(a) $f(x) = 5^x - \log_5(\sin(5x+5)) + \frac{1}{5\sqrt[5]{x}} + \arcsin(5x^5) - \arctan(5)$

$$f'(x) = 5^x \cdot \ln(5) - \frac{1}{\sin(5x+5)\ \ln(5)} \cdot \cos(5x+5) \cdot 5 + \frac{1}{5} \cdot \frac{-1}{5} x^{-6/5}$$

$$+ \frac{1}{\sqrt{1-(5x^5)^2}} \cdot 25x^4 + 0$$

(b) $g(x) = \arctan\left(\ln\left(e^{x\sec(3x)}\right)\right)$

$$g'(x) = \frac{1}{1+\left[\ln\left(e^{x\sec(3x)}\right)\right]^2} \cdot \frac{1}{e^{x\sec(3x)}} \cdot e^{x\sec(3x)} \cdot \left[x \cdot \sec(3x)\tan(3x)\cdot 3 + \sec(3x)\cdot 1\right]$$

(c) $g(x) = \arctan\left(\ln\left(e^{x\sec(3x)}\right)\right) = \arctan(x \cdot \sec(3x))$

$$g'(x) = \frac{1}{1+[x\sec(3x)]^2} \cdot \left[x \cdot \sec(3x)\tan(3x)\cdot(3) + \sec(3x)\cdot 1\right]$$

$\cos^2(y) = (\cos y)^2$

$(\sin x)^2$

2. Find $\frac{dy}{dx}$ for the following equations.

(a) $e^{x^4y^3} - \cos^2(y) = \sin^2(x) + \arcsin(y)$

$$e^{x^4y^3}\left(x^4\cdot 3y^2\cdot\frac{dy}{dx} + y^3\cdot 4x^3\right) - 2(\cos y)\cdot(-\sin y)\cdot\frac{dy}{dx} = 2(\sin x)\cdot\cos(x) + \frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx}$$

$$e^{x^4y^3}\cdot x^4\cdot 3y^2\,\frac{dy}{dx} + 2\sin(y)\cos(y)\cdot\frac{dy}{dx} - \frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx} = -e^{x^4y^3}\cdot y^3\cdot 4x^3 + 2\sin(x)\cos(x)$$

$$\boxed{\frac{dy}{dx} = \frac{-e^{x^4y^3}\cdot 4x^3y^3 + 2\sin(x)\cos(x)}{e^{x^4y^3}\cdot 3x^4y^2 + 2\sin(y)\cos(y) - \frac{1}{\sqrt{1-y^2}}}}$$

(b) $y = (\sqrt{x})^{\left(\frac{x}{\cot(x)}\right)}$

$$\ln(y) = \ln\left((\sqrt{x})^{\frac{x}{\cot(x)}}\right) = \frac{x}{\cot(x)}\cdot\ln(\sqrt{x}) = \frac{x}{\cot(x)}\ln\left(x^{1/2}\right) = \frac{x}{2\cot(x)}\cdot\frac{\ln(x)}{1}$$

$$\ln(y) = \frac{x\cdot\ln(x)}{2\cot(x)}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2\cot(x)\cdot\left[x\cdot\frac{1}{x} + \ln(x)\cdot 1\right] - x\ln(x)\cdot 2\left(-\csc^2(x)\right)}{(2\cot(x))^2}$$

$$\frac{dy}{dx} = \frac{2\cot(x)\left[1+\ln x\right] + 2x\ln(x)\csc^2(x)}{(2\cot(x))^2}\cdot\sqrt{x}^{\left(\frac{x}{\cot(x)}\right)}$$

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Typo* Typo*

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$2x^2$

3. Find all ~~points~~ x-values on the curve $f(x) = \frac{1}{3}x^3 + \cancel{\frac{3}{2}}x^2 + \frac{7}{4}x - 2024$ where the tangent line to the curve is parallel to the line $\boldsymbol{r}(t) = \langle 17 - 4t, -13 + 5t \rangle$.

$$\vec{m} = \langle -4, 5 \rangle = \langle \Delta x, \Delta y \rangle$$

$$m = \frac{\Delta y}{\Delta x} = \frac{5}{-4}$$

$$f'(x) = x^2 + 4x + \frac{7}{4}$$

$$x^2 + 4x + \frac{7}{4} = \frac{-5}{4}$$

$$x^2 + 4x + \frac{7}{4} + \frac{5}{4} = 0$$

$$x^2 + 4x + \frac{12}{4} = 0$$

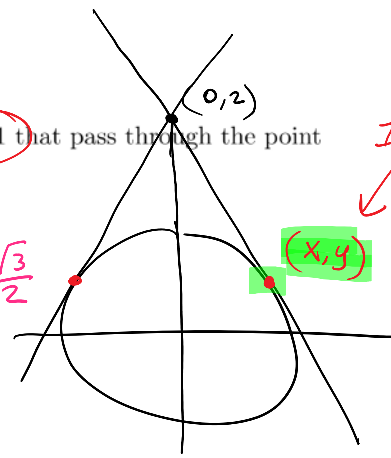
$$x^2 + 4x + 3 = 0$$

$$(x+3)(x+1) = 0$$

$$\boxed{x = -3 \qquad x = -1}$$

4. Find the equation of *both* tangent lines to the circle $x^2 + y^2 = 1$ that pass through the point $(0, 2)$.

Is on Circle



Slope: $\frac{y-2}{x-0} = \frac{y-2}{x}$

Derivative: $2x + 2y\frac{dy}{dx} = 0$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\frac{-x}{y} = \frac{y-2}{x}$$

$$-x^2 = y^2 - 2y$$

$$-(1-y^2) = y^2 - 2y$$

$$y^2 - 1 = y^2 - 2y$$

$$-1 = -2y \qquad y = \frac{1}{2}$$

$x^2 = 1 - y^2$

$x = \pm\frac{\sqrt{3}}{2}$

$$x^2 = 1 - \left(\tfrac{1}{2}\right)^2$$

$$x^2 = \frac{3}{4}$$

$$x = \frac{\pm\sqrt{3}}{2}$$

$$\frac{dy}{dx} = \frac{-\left(\sqrt{3}/2\right)}{1/2} = -\sqrt{3}$$

$$\frac{dy}{dx} = \frac{-\left(-\sqrt{3}/2\right)}{1/2} = \sqrt{3}$$

$$\boxed{\begin{aligned} y &= \sqrt{3}\cdot x + 2 \\ y &= -\sqrt{3}\cdot x + 2 \end{aligned}}$$

$\lim_{t \to}$ $\quad \dfrac{dy/dt}{dx/dt}$

5. Find the $t-$values corresponding to all points where the curve $x = 2t^3 - 6t, y = \left(t^2 + t - 6\right)^{13}$ has a horizontal or vertical tangent.

Horizontal: $\dfrac{dy}{dt} = 0$

$$\frac{dy}{dt} = 13\left(t^2 + t - 6\right)^{12} \cdot (2t+1) = 0$$

$$13\left[(t+3)(t-2)\right]^{12}(2t+1) = 0$$

$(t+3)^{12} = 0 \qquad (t-2)^{12} = 0 \qquad 2t+1 = 0$

$t+3 = 0 \qquad t-2 = 0 \qquad 2t = -1$

$$\boxed{t = -3 \qquad t = 2 \qquad t = -\tfrac{1}{2}}$$

Vertical: $\dfrac{dx}{dt} = 0$

$$\frac{dx}{dt} = 6t^2 - 6 = 0$$

$6t^2 = 6$

$t^2 = 1 \qquad 6(t^2-1) = 0$

$t = \pm 1 \qquad 6(t+1)(t-1) = 0$

$$\boxed{t = 1, \quad t = -1}$$

Typo*

$y = \frac{2}{3}t^3 + 3t^2$

6. Find the points where the curve $x = \ln t, \cancel{y = 2t^2 + 6t}$ has a horizontal tangent line.

Horizontal: $\dfrac{dy}{dt} = 0$

Domain: $\ln(t) \Rightarrow t > 0$

$$\frac{dy}{dt} = 2t^2 + 6t = 0$$

$$2t(t+3) = 0$$

$\cancel{t = 0} \qquad \cancel{t = -3}$

$\boxed{\text{No points w/ Horizontal Tangent Line}}$

$(t^2+5)^{1/2}$

7. Find a unit tangent vector to the curve $\boldsymbol{r}(t) = \left\langle \sqrt{t^2+5}, t \right\rangle$ when $t = 2$.

$\vec{r}\,'(t) = \langle \frac{1}{2}(t^2+5)^{-1/2} \cdot (2t), 1 \rangle$

$\vec{r}\,'(2) = \langle \frac{1}{2}(9)^{-1/2} \cdot (4), 1 \rangle$

$= \langle \frac{1}{2} \cdot \frac{1}{3} \cdot 4, 1 \rangle = \langle \frac{2}{3}, 1 \rangle$

Unit Tangent Vector

$$\boxed{\hat{r}\,'(2) = \frac{\langle \frac{2}{3}, 1 \rangle}{\sqrt{(\frac{2}{3})^2 + 1^2}}}$$

8. The height in meters of projectile shot vertically upward from a point 2 m above ground level with an initial velocity of 24.5 m/s is $h = 2 + 24.5t - 4.9t^2$ after t seconds. Potentially useful information for this question: $h(2.5) = 32.625$, $h(4) = 21.6$ and $h(-0.08) = h(5.08) = 0$ (approximately).

Velocity = 0

+ velocity

− velocity

height = 0

(a) What is the maximum height of the projectile?

Velocity: $v(t) = h'(t) = 24.5 - 9.8t = 0$

$-9.8t = -24.5$

$t = \frac{5}{2} = 2.5\,s$

$$\boxed{h(2.5) = 32.625\ m}$$

(b) What is the velocity of the projectile when it hits the ground?

$-4.9t^2 + 24.5t + 2 = 0$

$t = \frac{-24.5 \pm \sqrt{(24.5)^2 - 4(-4.9)(2)}}{2(-4.9)}$

$\cancel{t = -0.08}$

$t = 5.08$

$v(5.08) = 24.5 - 9.8(5.08)$

$= \boxed{-25.784\ m/s}$

(c) What is the total distance covered by the object after 4 seconds?

$h(0) = 2$

$h(4) = 21.6$

t	$h(t)$
0	2
2.5	32.625
4	21.6

30.625 m

+ 11.025 m

$\boxed{41.65\ m}$

9. Consider the piecewise function below.

$$f(x) = \begin{cases} x^2 + x + 2 & \text{if} \quad x \le -1 \\ -x + 1 & \text{if} \quad -1 < x \le 0 \\ -x + 5 & \text{if} \quad 0 < x < 2 \\ \sqrt{x+7} & \text{if} \quad 2 \le x \end{cases}$$

$(x+7)^{1/2}$

(a) Determine $f'(x)$ for all x−values other than $x = -1, x = 0$, and $x = 2$.

$$f'(x) = \begin{cases} 2x + 1 & \text{if} \quad x < -1 \\ -1 & \text{if} \quad -1 < x < 0 \\ -1 & \text{if} \quad 0 < x < 2 \\ \frac{1}{2}(x+7)^{-1/2} & \text{if} \quad 2 < x \end{cases}$$

(b) Is $f(x)$ differentiable at $x = -1$?

$\boxed{\text{Yes}}$

$f(x)$ Continuous @ $x = -1$?

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x^2 + x + 2 = 1 - 1 + 2 = 2$ ✓

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} -x + 1 = -(-1) + 1 = 1 + 1 = 2$

$f'(x)$ Continuous @ $x = -1$?

$\lim_{x \to -1^-} f'(x) = \lim_{x \to -1^-} 2x + 1 = -1$ ✓

$\lim_{x \to -1^+} f'(x) = \lim_{x \to -1^+} -1 = -1$

(c) Is $f(x)$ differentiable at $x = 0$?

$\boxed{\text{No}}$

$f(x)$ @ $x = 0$

LHL: 1

RHL: 5

$f(x)$ not continuos @ $x = 0$

-1 ✓

-1

(d) Is $f(x)$ differentiable at $x = 2$?

$\boxed{\text{No}}$

$f(x)$ @ $x = 2$

LHL: 3 ✓

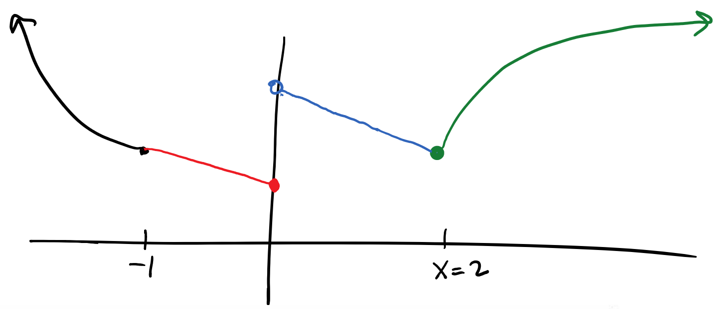
RHL: 3

$f'(x)$ @ $x = 2$

LHL: -1

RHL: $\frac{1}{6}$ ✗

(e) Draw a rough sketch of $f(x)$.



$y =$ population $\qquad \frac{dy}{dt} = k \cdot y$

$\frac{dy}{dt} =$ growth over time

10. A population grows at a rate proportional to its size. If the population is 8000 in 1990 and 20000 in 2001, ~~in~~ in what year will the population reach 40000?

$t=0$ (1990), $t=11$ (2001)

$$y = y_0 e^{kt}$$

$$8000 = y_0 e^{k(0)}$$

$$y_0 = 8000$$

$$20000 = 8000 e^{k(11)}$$

$$\frac{20}{8} = e^{11k}$$

$$\ln\left(\frac{5}{2}\right) = 11k$$

$$k = \frac{1}{11}\ln\left(\frac{5}{2}\right)$$

$$y = y_0 e^{kt}$$

$$40000 = 8000 e^{\frac{1}{11}\ln\left(\frac{5}{2}\right)\cdot t}$$

$$\frac{40}{8} = e^{\frac{1}{11}\ln\left(\frac{5}{2}\right)\cdot t}$$

$$\ln(5) = \frac{1}{11}\ln\left(\frac{5}{2}\right)\cdot t$$

$$\boxed{t = \frac{11\cdot\ln(5)}{\ln(5/2)}} \approx 19.3$$

$$\boxed{2009}$$

11. The half life of a substance is 60 years.

(a) How long will it take the substance to decay to 20% of its original amount?

$$\frac{1}{2}y_0 = y_0 e^{k\cdot(60)}$$

$$\frac{1}{2} = e^{60k}$$

$$\ln(1/2) = 60k$$

$$k = \frac{\ln(1/2)}{60}$$

$$0.2y_0 = y_0 e^{\frac{\ln(1/2)}{60}\cdot t}$$

$$\frac{1}{5} = e^{\frac{\ln(1/2)}{60}\cdot t}$$

$$\ln(1/5) = \frac{\ln(1/2)}{60}\cdot t$$

$$\boxed{t = \frac{60\ln(1/5)}{\ln(1/2)}} = \frac{60\ln(5)}{\ln(2)}$$

$$\ln(1/5) = \ln(5^{-1}) = -\ln(5)$$

$$\ln(1/2) = -\ln(2)$$

(b) How long will it take the substance to decay to $\frac{1}{16}$ of its original amount?

$$1 \xrightarrow{\text{60 years}} \frac{1}{2} \xrightarrow{\text{60 years}} \frac{1}{4} \xrightarrow{\text{60 years}} \frac{1}{8} \xrightarrow{\text{60 years}} \frac{1}{16}$$

Same Approach

$$\frac{60\ln(1/16)}{\ln(1/2)} = \frac{60\ln(16)}{\ln(2)} = \frac{60\ln(2^4)}{\ln(2)} = \frac{60\cdot 4\ln(2)}{\ln(2)} = 60\cdot 4 = \boxed{240 \text{ years}}$$

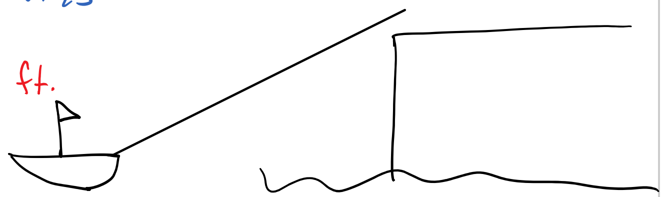
12. During a low tide, a boat is being towed to the dock by a rope. The rope is pulled from a position that is 7-ft above the water level at a rate of 2 ft/s. How fast is the boat approaching the ladder at the base of the dock when the boat is 24 ft from the ladder?

$x^2 + 7^2 = z^2$

$\frac{dz}{dt} = -2$ ft./s

$\frac{d}{dt}\left[x^2 + 49\right] = \frac{d}{dt}\left[z^2\right]$

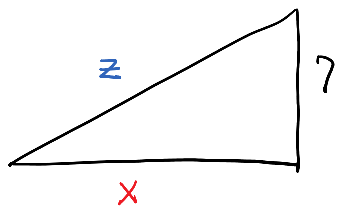
$X = 24$ ft.



ft, ft./s, ft, ft./s

$2x \cdot \frac{dx}{dt} = 2z \cdot \frac{dz}{dt}$

$z = 25$ ft.



$24 \cdot \frac{dx}{dt} = 25(-2)$

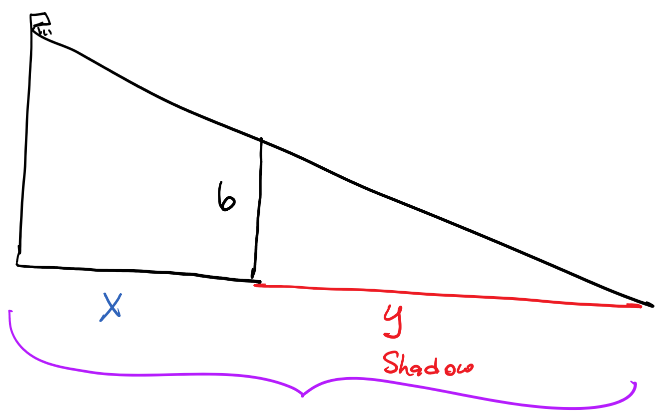
$\frac{dx}{dt} = \frac{-50}{24} = \boxed{\frac{-25}{12} \text{ ft./s}}$

$24^2 + 7^2 = z^2 \qquad z = 25$

13. A street light is mounted at the top of a 20-ft pole. A 6-ft man walks away from the pole with a speed of 5 ft/s. How fast is the tip of his shadow moving when he is 30 ft from the pole? Note: This question is trickier than it seems.

$\frac{y}{6} = \frac{x+y}{20}$

$\frac{dx}{dt} = 5$ ft./s



$20y = 6x + 6y$

$14y = 6x$

$y = \frac{3}{7}x$

$\frac{dy}{dt} = \frac{3}{7}\frac{dx}{dt}$

$\frac{dx}{dt} + \frac{dy}{dt} = 5 + \frac{15}{7} = \boxed{\frac{50}{7} \text{ ft./s}}$

$\frac{dy}{dt} = \frac{3}{7}(5) = \frac{15}{7}$ ft./s ?