1. Find the linear approximation for the function $f(x) = \sin x$ at $a = \pi/6$.

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$ linearization.

$f(x) = \sin x$	$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$
$f'(x) = \cos x$	$f'\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$$\sin x \approx f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right)$$

$$\boxed{\sin x \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)}$$ linear approximation.

2. Use differentials to approximate $(1.97)^6$.

$(1.97)^6$

$$f(a + \Delta x) \approx f(a) + f'(a)\Delta x$$

$f(x) = x^6$, $a = 2$, $\Delta x = 1.97 - 2 = -0.03$

$f(x) = x^6$	$f(2) = 2^6 = 64$
$f'(x) = 6x^5$	$f'(2) = 6 \cdot 2^5 = 64 \cdot 3 = 192$

$$(1.97)^6 \approx f(2) + f'(2)\Delta x$$

$$= 64 + 192(-0.03) \approx 63.9424$$

$r = 14, \ \Delta r = 0.5$

3. The radius of a sphere was measured to be 14 cm with a possible error of 0.5 cm.

(a) Use differentials to estimate the maximum error in the calculated surface area.

$$S.A. = 4\pi r^2$$

$$\Delta(S.A.) = (4\pi r^2)'_r \, \Delta r$$

$$\Delta(S.A.) = 8\pi r \, \Delta r$$

$$= 8\pi(14)\tfrac{1}{2} = \boxed{56\pi}$$

(b) What is the relative error?

$$\frac{\Delta(S.A.)}{S.A.} = \frac{\cancel{28\pi r}\,\Delta r}{\cancel{4\pi} r^{\cancel{2}}} = \frac{2\Delta r}{r} = \frac{1}{14} \approx \boxed{0.071}$$

percentage error: $0.071 \cdot 100\% = \boxed{7.1\%}$

(c) Use differentials to estimate the maximum error in the calculated volume.

$$V = \frac{4}{3}\pi r^3$$

$$\Delta V = \frac{4}{3}\pi (r^3)' \, \Delta r$$

$$\Delta V = 4\pi r^2 \, \Delta r$$

$$\Delta V = 4\pi(14)^2(0.5)$$

$$= 2\pi(196)$$

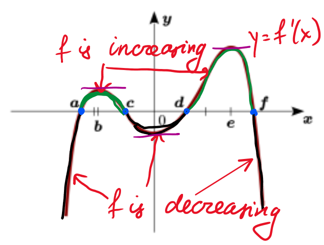
$$= 392\pi$$

(d) What is the relative error?

$$\frac{\Delta V}{V} = \frac{\cancel{4\pi}\,\cancel{r^2}\,\Delta r}{\frac{\cancel{4}}{3}\cancel{\pi} r^{\cancel{3}}} = \frac{3\Delta r}{r} = \frac{3\cdot(0.5)}{14} = \frac{3}{28} \approx \boxed{0.107}$$

percentage error $(0.107)\ 100\% = \boxed{10.7\%}$

4. The graph of the derivative, $f'(x)$, is shown below. Use the graph to answer these questions.



(a) On what intervals is f increasing? decreasing?

increasing $\Rightarrow f'(x) > 0 \Rightarrow (a, c) \cup (d, f)$

decreasing $\Rightarrow f'(x) < 0 \Rightarrow (-\infty, a) \cup (c, d) \cup (f, \infty)$

(b) On what intervals is f concave up? concave down?

f is concave up when $f''(x) > 0$ or $f'(x)$ is increasing $(-\infty, b) \cup (0, e)$

f is concave down when $f''(x) < 0$ or $f'(x)$ is decreasing $(b, 0) \cup (e, \infty)$

(c) At what values of x does f have a local maximum or minimum?

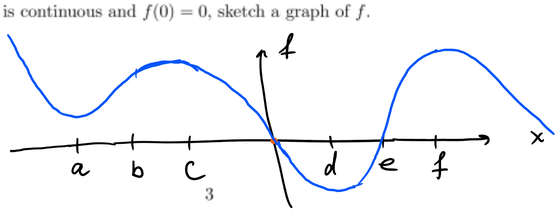
$x = a$ local min, $x = c$ local max, $x = d$ local min, $f = x$ local max

(d) At what values of x does f have an inflection point?

inflection point is when $f''(x) = 0$ or $f'(x)$ has a horizontal tangent

$x = b$, $x = 0$, $x = e$.

(e) Assuming that f is continuous and $f(0) = 0$, sketch a graph of f.

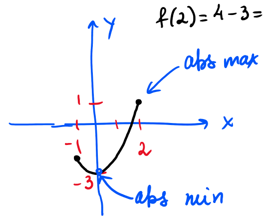


5. Find all absolute and local extrema for the following functions by graphing.

(a) $f(x) = x^2 - 3$, $-1 \le x \le 2$.

$x^2 - 3 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3} \approx 1.7$.

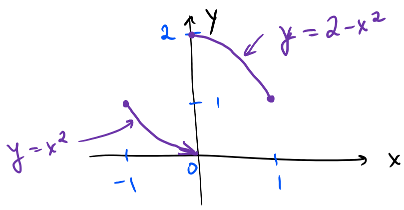
$f(2) = 4 - 3 = 1$



abs min value $f(0) = -3$

abs max value $f(2) = 1$

(b) $f(x) = \begin{cases} x^2, & \text{if } -1 \le x < 0 \\ 2 - x^2, & \text{if } 0 \le x \le 1 \end{cases}$



abs max value is $f(0) = 2$

no abs min value

critical number $\Longrightarrow$ $f'(x)=0$ or $f'(x)$ DNE.

6. Find all critical numbers for the following functions.

(a) $f(x) = \sqrt[3]{x}(x-1)^2 = x^{1/3}(x-1)^2$

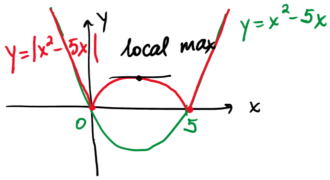
$$f'(x) = \frac{1}{3}x^{-2/3}(x-1)^2 + x^{1/2}\,2(x-1) = \frac{(x-1)^2}{3x^{2/3}} + 2x^{1/2}(x-1)$$

$$= \frac{(x-1)^2 + 6x(x-1)}{3x^{2/3}} = \frac{(x-1)[x-1+6x]}{3x^{2/3}} = \frac{(x-1)(7x-1)}{3x^{2/3}} = 0$$

$\boxed{x=1,\ x=1/7}$ $\leftarrow$ $f'(x)=0$

$\boxed{x=0}$ $\leftarrow$ $f'(x)$ DNE

(b) $f(x) = |x^2 - 5x| = \begin{cases} x^2-5x, & x^2-5x \geq 0 \\ -(x^2-5x), & x^2-5x < 0 \end{cases}$



$\boxed{x=0}$, $\boxed{x=5}$ $\leftarrow$ $f'(x)$ DNE

$f(x) = (x^2-5x) \Longrightarrow f'(x) = 2x-5 = 0 \Longrightarrow \boxed{x=\frac{5}{2}}$

(c) $f(x) = xe^{-2x}$

$$f'(x) = e^{-2x} - 2xe^{-2x}$$

$$= e^{-2x}(1-2x) = 0$$

since $e^{-2x} \neq 0$, then $1-2x=0$ or $\boxed{x=\frac{1}{2}}$

7. Find the absolute maximum value and absolute minimum value of the given function on the given interval.

(a) $f(x) = x^3 - 12x + 1$, $[-3, 5]$

End points $f(-3) = (-3)^3 - 12(-3) + 1 = -27 + 36 + 1 = \boxed{10}$

$f(5) = 5^3 - 12(5) + 1 = 125 - 60 + 1 = \boxed{66}$ abs max value

critical points inside the interval $(-3, 5)$

$f'(x) = 3x^2 - 12 = 0 \Rightarrow x^2 = \frac{12}{3} \Rightarrow x^2 = 4$, $\boxed{x = \pm 2.}$ critical points.

$f(-2) = (-2)^3 - 12(-2) + 1 = -8 + 24 + 1 = \boxed{17}$

$f(2) = 2^3 - 12(2) + 1 = 8 - 24 + 1 = \boxed{-15}$ abs min value

(b) $f(x) = \frac{\ln x}{x}$, $[1, 3]$

End points:

$f(1) = \frac{\ln 1}{1} = \boxed{0}$ abs min value

$f(3) = \frac{\ln 3}{3} \approx \boxed{0.366}$

Critical points inside the interval $(1, 3)$

$$f'(x) = \frac{\frac{1}{x} \cdot x - 1 \cdot \ln x}{x^2} = \frac{1 - \ln x}{x^2} = 0 \Rightarrow \ln x = 1$$

$$\boxed{x = e} \approx 2.7$$

$f(e) = \frac{\ln e}{e} = \frac{1}{e} \approx \boxed{0.368}$ abs max value

(c) $f(t) = 16\cos t + 8\sin 2t,\ \left[0, \frac{\pi}{2}\right]$

End points $\quad f(0) = 16\cancel{\cos 0}^{1} + 8\cancel{\sin 0}^{0} = \boxed{16}$

$f\left(\frac{\pi}{2}\right) = 16\cancel{\cos\frac{\pi}{2}}^{0} + 8\cancel{\sin\pi}^{0} = \boxed{0}$ abs min value

Critical points inside the interval $\left(0, \frac{\pi}{2}\right)$

$f'(t) = -16\sin t + 8(2)\cos 2t$

$= \dfrac{-16\sin t + 16\cos 2t}{-16} = \dfrac{0}{-16} \Rightarrow \sin t - \cos 2t = 0$

$\cos 2t = 1 - 2\sin^2 t$

$\sin t - (1 - 2\sin^2 t) = 0$

$2\sin^2 t + \sin t - 1 = 0$

substitution: $\quad \sin t = z, \quad |z| \le 1$

$2z^2 + z - 1 = 0$

$(2z - 1)(z + 1) = 0$

$2z - 1 = 0 \quad$ or $\quad z + 1 = 0$

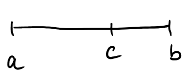
$\boxed{z = \frac{1}{2}} \qquad \boxed{z = -1}$

$\sin t = \frac{1}{2} \qquad \sin t = -1$

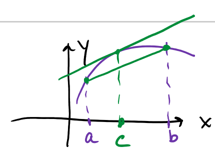
$\boxed{t = \frac{\pi}{6}} \qquad \cancel{t = \frac{3\pi}{2}}$ (out of range)

$f(t) = 16\cos t + 8\sin 2t$

$f\left(\frac{\pi}{6}\right) = 16\cancel{\cos\frac{\pi}{6}}^{\sqrt{3}/2} + 8\cancel{\sin\frac{\pi}{3}}^{\sqrt{3}/2} = 16\cdot\frac{\sqrt{3}}{2} + 8\cdot\frac{\sqrt{3}}{2} = 8\sqrt{3} + 4\sqrt{3} = 12\sqrt{3} \approx \boxed{20.78}$ abs max value



$\dfrac{f(b) - f(a)}{b - a} = f'(c)$



8. Find the number(s) c that satisfies the conclusion of the Mean Value Theorem on the given interval.

(a) $f(x) = 2x^3 + 1,\ [1, 2]$

$f'(x) = 6x^2$

$\dfrac{f(2) - f(1)}{2 - 1} = 6c^2 \qquad \left[\dfrac{f(2) - f(1)}{2 - 1} = f'(c)\right]$

$$\frac{2(2^3)+\cancel{1}-(2+\cancel{1})}{1} = 6c^2$$

$16-2=6c^2$ or $6c^2=14$ or $c^2=\frac{14}{6}=\frac{7}{3}$

$c=\pm\sqrt{\frac{7}{3}} \approx \pm 1.53$

$c=-1.53$ not in the interval

$$\boxed{c=\sqrt{\frac{7}{3}}}$$

(b) $f(x)=\ln x$, $[1,4]$

Find c such that $\frac{f(4)-f(1)}{4-1} = f'(c)$

$f'(x)=\frac{1}{x}$, $f'(c)=\frac{1}{c}$

$\frac{\ln 4-\cancel{\ln 1}^{\,0}}{4-1} = \frac{1}{c}$ or $\frac{\ln 4}{3}=\frac{1}{c} \Rightarrow \boxed{c=\frac{3}{\ln 4}}$

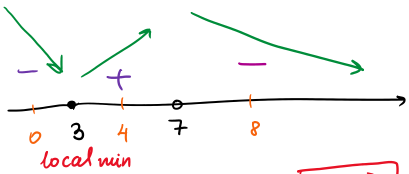
9. Suppose the function $f(x)$ has a domain of all real numbers except $x = 7$. The first derivative of $f(x)$ is

$$f'(x) = \frac{-2(x-3)e^{8x}}{(x-7)^7}.$$

(a) Find the interval(s) where $f(x)$ is increasing.

$f'(x) > 0$ or $\frac{-2(x-3)e^{8x}}{(x-7)^7} > 0$

since $e^{8x} > 0$ then $\frac{-2(x-3)}{(x-7)^7} > 0$



$\frac{-2(8-3)}{(8-3)^7} < 0$

$\frac{-2(0-3)}{(0-3)^7} < 0$

$\frac{-2(4-3)}{(4-7)^7} > 0$

increases on $\boxed{(3,7)}$

(b) Find the interval(s) where $f(x)$ is decreasing.

$f'(x) < 0$ $\boxed{(-\infty, 3) \cup (7, \infty)}$ decreases

(c) Find the x-coordinates of all local extrema on the graph of $f(x)$.

$x = 3$, $x = 7$.

10. Suppose the function $g(x)$ has a domain of all real numbers. The second derivative of $g(x)$ is

$$g''(x) = (x-2)^5(x+4)(x+8)^4.$$

(a) Find the interval(s) where $g(x)$ is concave up.

concave up $\Longrightarrow$ $g''(x) > 0$

always positive

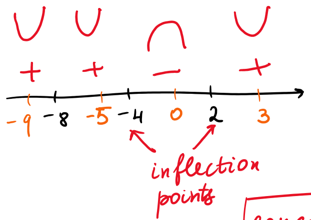
$g(x) = (x-2)^5(x+4)(x+8)^4 > 0$

$g(-9) = (-9-2)^5(-9+4)(-9+8)^4 > 0$

$g(-5) = (-5-2)^5(-5+4)(-5+8)^4 > 0$

$g(0) = (0-2)^5(0+4)(0+8)^4 < 0$

$g(3) = (3-2)^5(3+4)(3+8)^4 > 0$



$\boxed{\text{concave up on } (-\infty, -4) \cup (2, \infty)}$

(b) Find the interval(s) where $g(x)$ is concave down.

concave down $\Longrightarrow$ $g''(x) < 0$

$\boxed{\text{concave down on } (-4, 2)}$

(c) Find the x-coordinates of the inflection points of $g(x)$.

$x = -4$

$x = 2$