Review of Sections 3.7, 3.8, 3.9

1. The ball is tossed into the air. Its position at time t is given by $\mathbf{r}(t) = < 5t, \overbrace{100t - 16t^2}^{y(t)} >$.
   (a) Find the velocity and the speed of the ball when $t = 2$.
   (b) How high does the ball go?
   (c) With what speed does the ball hit the ground?

$\vec{v}(t) = \vec{r}\,'(t) = < 5, 100 - 32t >$

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = < 0, -32 >$

(a) velocity $\vec{v}(2) = < 5, 100 - 32\cdot 2 > = < 5, 100 - 64 > = \boxed{< 5, 36 >}$

speed $s(2) = |\vec{v}(2)| = \sqrt{5^2 + 36^2} = \boxed{\sqrt{1321}}$

(b) the vertical component of $\vec{r}(t)$ should be zero.

$100 - 32t = 0 \Rightarrow t_{max} = \frac{100}{32} = \frac{4\cdot 25}{8\cdot 4} = \frac{25}{8}$ (sec).

$h_{max} = y(t_{max}) = 100 \cdot \frac{25}{8} - 16 \cdot \left(\frac{25}{8}\right)^2 = \frac{625}{2} - 16 \cdot \frac{625}{64} = \ldots = \boxed{\frac{625}{4}}$

(c) find t such that $y(t) = 0$

$100t - 16t^2 = 0$ or $t(100 - 16t) = 0$

$t = 0$ or $100 - 16t = 0$

$t = \frac{100}{16} = \frac{25}{4}$

$\vec{v}\left(\frac{25}{4}\right) = < 5, 100 - 32 \cdot \frac{25}{4} > = < 5, 100 - 200 > = < 5, -100 >$

$\left|\vec{v}\left(\frac{25}{4}\right)\right| = \text{speed}\left(\frac{25}{4}\right) = \sqrt{5^2 + (-100)^2} = \boxed{\sqrt{10025}}$

2. If $\mathbf{r}(t) = < t^3, t^2 >$ represents the position of a particle at time t, find the angle between the velocity and the acceleration vector at time $t = 1$.

$\vec{r}(t) = < t^3, t^2 >$ position

$\vec{v}(t) = \vec{r}\,'(t) = < 3t^2, 2t >$ velocity

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = < 6t, 2 >$ acceleration.

Find an angle between $\vec{v}(1)$ and $\vec{a}(1)$

$\vec{v}(1) = < 3, 2 >$

$\vec{a}(1) = < 6, 2 >$

$$\cos\theta = \frac{\vec{v}(1) \cdot \vec{a}(1)}{|\vec{v}(1)| \cdot |\vec{a}(1)|} = \frac{< 3,2 > \cdot < 6,2 >}{\sqrt{9+4} \cdot \sqrt{36+4}} = \frac{18+4}{\sqrt{13}\sqrt{40}} = \frac{22}{\sqrt{13}(2\sqrt{10})} = \frac{11}{\sqrt{130}}$$

$\sqrt{40} = \sqrt{4 \cdot 10} = 2\sqrt{10}$

$\theta = \arccos \frac{11}{\sqrt{130}} \approx 15°$

3. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after 5 sec.

$v = 60$ cm/s.

$r(t) = 60t$ (radius of the circle).

[area within the circle] $\rightarrow A(t) = \pi r^2 = \pi(60t)^2$

$\frac{d}{dt} A(t) = \frac{d}{dt} 3600\pi t^2$

[rate of change of the area] $\rightarrow \frac{dA}{dt} = 3600\pi(2t)$

$= 7200\pi t$

after 5 sec $(t=5)$ $\frac{dA}{dt} = 7{,}200\pi(5) = \boxed{36{,}000\pi \text{ (cm}^2\text{/s)}}$

4. If a ball is thrown vertically upward with a velocity of 144 ft/s, then its height after t seconds is $s = 144t - 16t^2$.

(a) What is the maximum height reached by the ball?
(b) What is the velocity of the ball when it is 320 ft above the ground on his way up?
(c) What is the velocity of the ball when it is 320 ft above the ground on his way down?
(d) When will the ball hit the ground?
(e) With what velocity does the ball hit the ground?

$s(t) = 144t - 16t^2$.

(a) h_{max} is reached when $v(t) = 0$.

$v(t) = s'(t) = 144 - 32t = 0$

$t_{max} = \frac{144}{32} = \frac{9}{2}$

$h_{max} = s(t_{max}) = 144\left(\frac{9}{2}\right) - 16\left(\frac{9}{2}\right)^2 = 648 - 324 = 324$ (ft)

(b) Find t such that $\boxed{s(t) = 320}$

$144t - 16t^2 = 320$ or $\frac{16t^2 - 144t + 320}{16} = \frac{0}{16}$

$t^2 - 9t + 20 = 0$

$(t-4)(t-5) = 0$

$t_1 = 4$, $t_2 = 5$

(way up) (way down)

(b) $v(4) = 144 - 32(4) = 16$ (ft/s)

(c) $v(5) = 144 - 32(5) = -16$ (ft/s)

or $v(5) = 16$ (ft/s) the velocity vector is directed downward.

(d) The ball will hit the ground when $s(t) = 0$.

$144t - 16t^2 = 0$

$t(144 - 16t) = 0$

$t_1 = 0$, $t_2 = \frac{144}{16} = \boxed{9 \text{ (s)}}$

(e) The velocity is $v(9) = 144 - 32(9) = -144$ or $|v(9)| = \boxed{144 \text{ (ft/s)}}$

5. A bacteria culture starts with 1000 bacteria and the growth rate is proportional to the number of bacteria. After 2 h the population is 9000.

(a) Find an expression for the number of bacteria after t hours.

(b) Find the number of bacteria after 3 h.

(c) In what period of time does the number of bacteria double?

If $p(t)$ represents the population of bacteria, then $\boxed{p(t) = p_0 e^{kt}}$, k is an unknown const, p_0 is the initial population.

$p_0 = 1{,}000$

$p(2) = 9{,}000$

$p(t) = 1{,}000\, e^{kt}$

population after 2h $\longrightarrow$ $p(2) = \underbrace{\cancel{1{,}000}\, e^{2k} = 9{,}\cancel{000}}_{\text{solve for } k}$

$\ln(e^{2k}) = \ln 9$

$2k \cancel{\ln e} = \ln 9 \Rightarrow 2k = \ln 9$

$\boxed{k = \frac{\ln 9}{2}}$

update the formula $p(t) = 1{,}000\, e^{t \frac{\ln 9}{2}}$

$= 1{,}000 \left(\cancel{e^{\ln 9}}\right)^{\frac{t}{2}}$

$= 1{,}000 (9)^{t/2}$

$= 1{,}000 \cdot (9^{1/2})^t$

$\boxed{p(t) = 1{,}000 (3^t)}$

The number of bacteria after 3 hours is $p(3) = 1{,}000 \cdot 3^3 = \boxed{27{,}000}$

(c) Find t such that $p(t) = 2p_0 = 2{,}000$

solve for t $\quad \frac{2{,}000}{1{,}000} = \frac{(1{,}000)(3^t)}{1{,}000} \Rightarrow \ln 3^t = \ln 2$

$t \ln 3 = \ln 2$

$t = \frac{\ln 2}{\ln 3} \approx \boxed{0.63 (h)}$

6. An isotope of strontium, Sr^{90}, has a half-life of 25 years.

(a) Find the mass of Sr^{90} that remains from a sample of 18 mg after t years.

(b) How long will it take for the mass to decay to 2 mg?

If $m(t)$ is the mass of the sample after t years.

$\boxed{m(25) = \frac{1}{2} m(0)}$ $\boxed{m(t) = m(0)e^{kt}}$

$m(25) = \frac{m(0)e^{25k}}{m(0)} = \frac{\frac{1}{2}m(0)}{m(0)}$

$\ln e^{25k} = \ln \frac{1}{2}$

$25k \ln e = \ln \frac{1}{2} = -\ln 2$

$25k = -\ln 2$ or $\boxed{k = -\frac{\ln 2}{25}}$

$m(t) = m(0)\, e^{-\frac{t\ln 2}{25}}$

$= m(0)\left(e^{\ln 2}\right)^{-\frac{t}{25}}$

$\boxed{m(t) = m(0)\left(2^{-\frac{t}{25}}\right)}$

(a) $m(0) = 18 \Rightarrow$ $\boxed{m(t) = 18 \cdot \left(2^{-\frac{t}{25}}\right)}$

(b) Find t such that $m(t) = 2$

$\frac{2}{18} = \frac{18}{18} \cdot \left(2^{-\frac{t}{25}}\right)$

$-\ln 9 = \ln\left(\frac{1}{9}\right) = \ln\left(2^{-\frac{t}{25}}\right)$

$+\frac{t}{25}\ln 2 = +\ln 9 \Rightarrow t = \frac{25\ln 9}{\ln 2} \approx \boxed{79 \text{ (years)}}$

7. A cup of coffee has a temperature of 200°F and is in a room that has a temperature of 70°F. After 10 min the temperature of the coffee is 150°F.

The Newton's Law of cooling : the rate of chang of the temp of the object is proportinal to the difference of the current temp and the temp of the medium.

$T(t)$ is the temp of the object, then

$$\boxed{\frac{dT}{dt} = k(\underbrace{T-M}_{u(t)})}$$, M is the temp of the medium. [room temp]

$M = 70$

$T(0) = 200$, $T(10) = 150$

(a) find the temp of coffee after 15 min

substitution $u(t) = T(t) - 70$, then $T(t) = u(t) + 70$

$$\frac{dT}{dt} = \frac{du}{dt}$$

$$\boxed{\frac{du}{dt} = ku}, \quad u(0) = T(0) - 70 = 200 - 70$$

$$\boxed{u(0) = 130}$$

$$u(t) = u(0)e^{kt} \quad \text{or} \quad u(t) = 130e^{kt}$$

switch back to $T(t)$

$$T(t) - 70 = 130e^{kt}$$

$$T(t) = 70 + 130e^{kt}$$

$$T(10) = \underbrace{150}_{} \Rightarrow T(10) = \underbrace{70 + 130e^{10k}}_{}$$

$$\frac{150}{10} = \frac{70 + 130e^{10k}}{10} \Rightarrow 13e^{10k} = 15 - 7$$

$$13e^{10k} = 8$$

$$\ln e^{10k} = \ln \frac{8}{13}$$

$$10k = \ln \frac{8}{13}$$

$$\boxed{k = \frac{1}{10} \ln \frac{8}{13}}$$

$$T(t) = 70 + 130e^{\frac{t}{10} \ln \frac{8}{13}}$$

$$T(15) = 70 + 130e^{\frac{15}{10} \ln \frac{8}{13}} \approx \boxed{133\ ^\circ F}$$

(b) When will it cool to 100° F?

Find t such that $T(t) = 100$ or $\dfrac{70 + 130e^{\frac{t}{10} \ln \frac{8}{13}}}{10} = \dfrac{100}{10}$

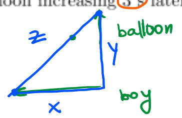
$$13e^{\frac{t}{10} \ln \frac{8}{13}} = 10 - 7$$

$$13e^{\frac{t}{10} \ln \frac{8}{13}} = 3$$

$$\ln e^{\frac{t}{10} \ln \frac{8}{13}} = \ln \frac{3}{13} \Rightarrow \frac{t}{10} \ln \frac{8}{13} = \ln \frac{3}{13}$$

$$t = \frac{10 \ln 3/13}{\ln 8/13} \approx \boxed{30\ (\text{min})}$$

8. A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 s later?



z is the distance between the boy and the balloon.

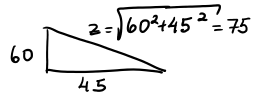
$$\boxed{z^2 = x^2 + y^2}$$

$\frac{dz}{dt} = ?$ | $\frac{dy}{dt} = 5$, $\frac{dx}{dt} = 15$

$$\frac{d}{dt}(z^2) = \frac{d}{dt}(x^2+y^2) \quad \text{(implicit differentiation)}$$

$$\frac{2z\frac{dz}{dt}}{2} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2}$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\cancel{\frac{dx}{dt}}^{15} + y\cancel{\frac{dy}{dt}}^{5}\right)$$



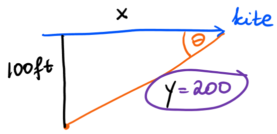
$x = 3 \cdot 15 = 45$ (distance travelled by the boy)

$y = 45 + 5 \cdot 3 = 60$ (the height of the balloon).

$z = \sqrt{x^2+y^2} = \sqrt{60^2+45^2} = 75$

$$\frac{dz}{dt} = \frac{1}{75}\left(45(15) + 60(5)\right) = \boxed{13 \text{ ft/s}}$$

9. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string have been let out?



Find $\frac{d\theta}{dt}$ when $y = 200$ ft $\qquad \frac{dx}{dt} = 8$

$$\tan\theta = \frac{100}{x}, \quad \frac{d}{dt}(\cot\theta) = \frac{d}{dt}\left(\frac{x}{100}\right)$$

$$-\csc^2\theta \frac{d\theta}{dt} = \frac{1}{100}\frac{dx}{dt}$$

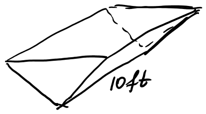
$$\frac{d\theta}{dt} = -\frac{\sin^2\theta}{100}\cancel{\frac{dx}{dt}}^{\,8}$$

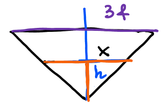
$$\sin\theta = \frac{100}{200} = \frac{1}{2}$$

$$\frac{d\theta}{dt} = -\frac{1}{4}\cdot\frac{1}{100}(8) = -\frac{1}{50} \text{ (rad/s)}$$

The angle decreases at a rate of $\frac{1}{50}$ rad/s

10. A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across the top and have a height of 1 ft. If the trough is filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 inches deep?





1 ft = h

h is water depth (we are measuring it from the bottom)

Volume of water $V = A(\text{side})(10)$

$V = \frac{10}{2} \times h$

$V = 5 \times h$

Find $\frac{dh}{dt}$ when $h = 6$ inches $= \frac{1}{2}$ ft

$\frac{dV}{dt} = 12\,(ft^3/min)$

Need a relation between x and h (express x in terms of h).

similar triangles: $\frac{x}{3} = \frac{h}{1}$ or $\boxed{x = 3h}$

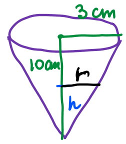
$V = 5(3h)h \Rightarrow \frac{d}{dt}V = 15\frac{d}{dt}(h^2)$

$\frac{dV}{dt} = 15(2h)\frac{dh}{dt}$ or $\frac{dh}{dt} = \frac{1}{30h}\frac{dV}{dt}$

$\left(h = 1/2,\ \frac{dV}{dt} = 12\right)$

$\frac{dh}{dt} = \frac{1}{15}(12) = \frac{12}{15} = \boxed{\frac{4}{5}\,(ft/min)}$

11. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If the water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?



h is water level

$\frac{dh}{dt} = ?$ when $h = 5$

$\frac{dV}{dt} = 2\ (cm^3/s)$

$V = \frac{1}{3}\pi r^2 h$

Express r in terms of h

From the similar triangles $\frac{r}{3} = \frac{h}{10} \Rightarrow r = \frac{3h}{10}$

$V = \frac{1}{3}\pi\left(\frac{3h}{10}\right)^2 h$

$= \frac{1}{3}\pi \frac{9h^3}{100}$ or $\frac{d}{dt}V = \frac{d}{dt}\frac{3\pi h^3}{100}$

$\frac{dV}{dt} = \frac{3\pi(3h^2)}{100}\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{100}{9\pi h^2}\frac{dV}{dt}$

$h = 5,\ \frac{dV}{dt} = 2$

$\frac{dh}{dt} = \frac{100}{9\pi(5)^2}(2) = \boxed{\frac{8}{9\pi}\ cm/min}$

Review for Exam 2.

$a(t) = v'(t) = 12t - 18$

1. An object is moving along a straight path. The position of the object at time t is given by

$$s(t) = 2t^3 - 9t^2 + 12t + 1 \implies v(t) = s'(t) = 6t^2 - 18t + 12$$

$v(t) = 6(t^2 - 3t + 2)$

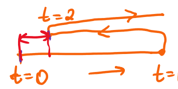
$= 6(t-1)(t-2)$

, where t is measured in seconds and $s(t)$ is measured in feet. Find

(a) the velocity and acceleration as functions of t.

(b) the acceleration when the velocity is zero.

(c) the total distance traveled in the first 2 seconds.



$s(2) - s(0) =$ displacement.

$v(t) = 6(t-1)(t-2)$

$v(0) > 0$

$v(1.5) = 6(1.5-1)(1.5-2) < 0$

$v(3) = 6(3-1)(3-2) > 0$

Total distance traveled $d = |s(1) - s(0)| + |s(2) - s(1)|$

$= |\underbrace{2-9+12+1}_{s(1)} \underbrace{-1}_{s(0)}| + |\underbrace{2\cdot 2^3 - 9\cdot 2^2 + 12\cdot 2+1}_{s(2)} - \underbrace{(2-9+12+1)}_{s(1)}|$

$= 5 + |5-6| = 5+1 = \boxed{6 \text{ (ft)}}$

2. At what point on the curve $f(x) = 36\sqrt{x}$ is the tangent line parallel to the line $\underbrace{9x - y + 2 = 0}_{\text{what is the slope?}}$?

$y = 9x + 2$

the slope is 9

Find x such that $f'(x) = 9$

slope $= f'(x) = \dfrac{36}{2\sqrt{x}} = 9 \implies \sqrt{x} = \dfrac{36}{18} = 2$

$x = 2^2 = 4.$

$f(4) = 36\sqrt{4} = \pm 36(2) = \pm 72$

$\boxed{(4, 72)}$ and $\boxed{(4, -72)}$

3. Find an equation of the tangent line to the curve $\boxed{2e^{xy} = x + y}$ at the point (0,2).

implicit differentiation.

$$2\frac{d}{dx}\left(e^{xy}\right) = \frac{d}{dx}(x+y)$$

$$2e^{xy}\frac{d}{dx}(xy) = 1 + \frac{dy}{dx}$$

product rule

$$2e^{xy}\left(y + x\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$$

$$2ye^{xy} + 2xe^{xy}\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(2xe^{xy} - 1\right) = 1 - 2ye^{xy}$$

$$\frac{dy}{dx} = \frac{1 - 2ye^{xy}}{2xe^{xy} - 1}$$

$$\left.\frac{dy}{dx}\right|_{(0,2)} \quad \frac{\text{plug in } x=0}{\text{and } y=2} \quad \frac{1 - 2(2)e^{0}}{0 \cdot e^{0} - 1} = 3$$

Tangent line

$$y - 2 = 3(x - 0)$$

$$\boxed{y = 3x + 2}$$

4. Find the values of a and b that make the function

$$f(x) = \begin{cases} ax^2 + x + 1, & \text{if } x \leq 1 \\ bx - 1, & \text{if } x > 1 \end{cases}$$

differentiable everywhere. Find $f'(x)$.

$f(x)$ must be continuous: $a \cdot 1^2 + 1 + 1 = b(1) - 1$ or $a + 2 = b - 1$, $\boxed{a = b - 3}$

$f'(x)$ must be continuous as well

$$f'(x) = \begin{cases} 2ax+1, & x \leq 1 \\ b, & x > 1 \end{cases} \Rightarrow 2a(1)+1 = b, \quad \boxed{2a+1 = b}$$

$$\begin{cases} a = b-3 \\ b = 2a+1 \end{cases}$$ solve for a and b

$b = 2(b-3)+1$ or $b = 2b-6+1 \Rightarrow \boxed{b = 5}$, $\boxed{a = 5-3 = 2}$

$$f(x) = \begin{cases} 2x^2+x+1, & x \leq 1 \\ 5x-2, & x > 1 \end{cases}, \text{ and } f'(x) = \begin{cases} 2(2)x+1, & x \leq 1 \\ 5, & x > 1 \end{cases} \text{ or } \boxed{f'(x) = \begin{cases} 4x+1, & x \leq 1 \\ 5, & x > 1 \end{cases}}$$

5. If $f(x) = \sin(g(x))$, find $f'(2)$ given that $g(2) = \frac{\pi}{3}$ and $g'(2) = \frac{\pi}{4}$.

$f'(x) = \cos(g(x))\, g'(x)$

$f'(2) = \cos(g(2))\, g'(2)$

$= \left(\cancel{\cos\frac{\pi}{3}}^{1/2}\right) \frac{\pi}{4} = \boxed{\frac{\pi}{8}}$

6. Find the derivative

(a) $f(x) = x^2 \cot(3x)$

$$f'(x) = (x^2)' \cot(3x) + x^2(\cot 3x)'$$

$$= 2x \cot(3x) + x^2(-\csc^2 3x)(3x)'$$

$$= \boxed{2x \cot(3x) - 3x^2 \csc^2(3x)}$$

(b) $f(x) = \dfrac{e^{\sqrt{x}}}{x+\sqrt{x}}$

$$f'(x) = \frac{(e^{\sqrt{x}})'(x+\sqrt{x}) - (x+\sqrt{x})' e^{\sqrt{x}}}{(x+\sqrt{x})^2}$$

$$= \frac{e^{\sqrt{x}}(\sqrt{x})'(x+\sqrt{x}) - \left(1+\frac{1}{2\sqrt{x}}\right)e^{\sqrt{x}}}{(x+\sqrt{x})^2}$$

$$= \frac{e^{\sqrt{x}}\frac{1}{2\sqrt{x}}(x+\sqrt{x}) - \frac{2\sqrt{x}+1}{2\sqrt{x}}e^{\sqrt{x}}}{(x+\sqrt{x})^2}$$

$$= \frac{e^{\sqrt{x}}(x+\sqrt{x}) - (2\sqrt{x}+1)e^{\sqrt{x}}}{2\sqrt{x}(x+\sqrt{x})^2} = \boxed{\frac{e^{\sqrt{x}}(x-\sqrt{x}+1)}{2\sqrt{x}(x+\sqrt{x})^2}}$$

(c) $f(x) = \tan^3(3^{-x} + ex - x^e)$

$$f'(x) = 3\tan^2(3^{-x} + ex - x^e)\left(\tan(3^{-x} + ex - x^e)\right)'$$

$$= 3\tan^2(3^{-x} + ex - x^e)\sec^2(3^{-x} + ex - x^e)(3^{-x} + ex - x^e)'$$

$$= \boxed{3\tan^2(3^{-x} + ex - x^e)\sec^2(3^{-x} + ex - x^e)(-3^{-x}\ln 3 + e - ex^{e-1})}$$

$$[g(x)]^{f(x)} = e^{f(t)\ln g(x)}$$

(d) $f(x) = \left(\frac{x^3 + 3x}{x^2 - 4x + 1}\right)^{\arctan x}$

$$f(x) = e^{\arctan x \ln \frac{x^3+3x}{x^2-4x+1}}$$

$$f'(x) = e^{\arctan x \ln \frac{x^3+3x}{x^2-4x+1}}\left[\arctan x \ln \frac{x^3+3x}{x^2-4x+1}\right]'$$

$$= \left(\frac{x^3+3x}{x^2-4x+1}\right)^{\arctan x}\left[\arctan x\left(\ln(x^3+3x) - \ln(x^2+4x+1)\right)\right]'$$

$$= \left(\frac{x^3+3x}{x^2-4x+1}\right)^{\arctan x}\left[\arctan(x)\ln(x^3+3x) - \arctan(x)\ln(x^2+4x+1)\right]'$$

$$= \left(\frac{x^3+3x}{x^2-4x+1}\right)^{\arctan x}\left[\frac{1}{1+x^2}\ln(x^3+3x) + \arctan(x)\cdot\frac{(x^3+3x)'}{x^3+3x} - \frac{1}{1+x^2}\ln(x^2+4x+1) - \arctan x\frac{(x^2+4x+1)'}{x^2+4x+1}\right]$$

$$= \left(\frac{x^3+3x}{x^2-4x+1}\right)^{\arctan x}\left[\frac{\ln(x^3+3x)}{1+x^2} + \arctan(x)\frac{3x^2+3}{x^3+3x} - \frac{\ln(x^2+4x+1)}{1+x^2} - \arctan(x)\frac{2x+4}{x^2+4x+1}\right]$$

7. The vector function $\mathbf{r}(t) = < t + e^{4t}, -t\cos(2t) >$, $0 \le t \le 2\pi$, represents the position of a particle at time t. Find the velocity and acceleration vectors of the object at $t = \frac{\pi}{4}$.

$$\vec{v}(t) = < 1 + 4e^{4t}, -\cos(2t) - t\,(-\sin 2t)(2) >$$

velocity : $\vec{r}'(t) = \vec{v}(t) = < 1 + 4e^{4t}, 2t\sin 2t - \cos 2t >$

$$\vec{v}\left(\frac{\pi}{4}\right) = < 1 + 4e^{\frac{4\pi}{4}}, 2\cdot\frac{\pi}{4}\sin\frac{2\pi}{4} - \cos\frac{2\pi}{4} >$$

$$= < 1 + 4e^{\pi}, \frac{\pi}{2}\cancel{\sin\frac{\pi}{2}}^{1} - \cancel{\cos\frac{\pi}{2}}^{0} >$$

$$\boxed{\vec{v}\left(\frac{\pi}{4}\right) = < 1 + 4e^{\pi}, \frac{\pi}{2} >}$$

acceleration : $\vec{a}(t) = \vec{v}'(t) = < 4e^{4t}(4), \underbrace{2\sin 2t} + 2t\cos(2t)(2) + \underbrace{2\sin 2t} >$

$$= < 16e^{4t}, 4\sin 2t + 4t\cos 2t >$$

$$\vec{a}\left(\frac{\pi}{4}\right) = < 16e^{\frac{4\pi}{4}}, 4\sin\frac{2\pi}{4} + 4\frac{\pi}{4}\cos\frac{2\pi}{4} > = < 16e^{\pi}, 4\cancel{\sin\frac{\pi}{2}}^{1} + \pi\cancel{\cos\frac{\pi}{2}}^{0} >$$

$$\boxed{\vec{a}\left(\frac{\pi}{2}\right) = < 16e^{\pi}, 4 >}$$

$x'(t) = 2t - 6$ $\quad y'(t) = 2t + 2$

8. At what point(s) does the curve parametrized by $\boxed{x = t^2 - 6t + 5}$ $\boxed{y = t^2 + 4t + 3}$ have a horizontal or vertical tangent?

slope of a tangent line is $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$

vertical tangent $\Rightarrow$ $x'(t) = 0$

$x'(t) = 2t - 6 = 0 \implies \boxed{t = 3}$ vertical

(slope = 0) horizontal tangent $\Rightarrow$ $y'(t) = 0$.

$y'(t) = 2t + 2 = 0 \Rightarrow \boxed{t = -1}$ horizontal