Math 151 - Week-In-Review 10

Topics for the week:

4.2 The Mean Value Theorem
4.3 How Derivatives Affect the Shape of a Graph

4.2 The Mean Value Theorem

1. Verify that the function, $f(x) = x^3 + 4x^2 - 5$, satisfies the three hypotheses of Rolle's theorem on the interval $[-4, 0]$. Then determine all numbers c that satisfy the conclusion of Rolle's theorem.

1. $f(x)$ is a polynomial and thus continuous on the closed interval $[-4,0]$.

2. $f'(x) = 3x^2 + 8x$ is also a polynomial and thus continuous on the closed interval $[-4,0]$ and $f(x)$ is differentiable on the open interval $(-4,0)$.

3. $f(-4) = (-4)^3 + 4(-4)^2 - 5 = -5$ so $f(-4) = f(0)$
$f(0) = (0)^3 + 4(0)^2 - 5 = -5$

$f(x)$ satisfies the conditions of Rolle's Theorem

Rolle's Theorem: there exists a real number c in $(-4,0)$ s.t. $f'(c) = 0$

$3x^2 + 8x = 0$
$x(3x+8) = 0$
so $x = 0$ or $3x + 8 = 0$
$3x = -8$
$x = -8/3$

$c = \frac{-8}{3}$ as 0 is not in $(-4,0)$

2. Does the function, $g(x) = \sqrt[3]{6 - x}$, satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 14]$. If it satisfies the hypotheses, compute all numbers c that satisfy the conclusion of the Mean Value Theorem.

1. $g(x) = (6-x)^{1/3}$ is an odd root function with domain $(-\infty, \infty)$, so $g(x)$ is continuous on $[0,14]$

2. $g'(x) = \frac{-1}{3(6-x)^{2/3}}$ and $g'(x)$ is not defined at $x = 6$

so $g(x)$ is not differentiable for all x in $(0,14)$.

$g(x)$ does not satisfy the conditions of the Mean Value Theorem.

3. Does the function, $h(x) = \frac{1}{2} \cdot 3^{x+1}$, satisfy the hypotheses of the Mean Value Theorem on the interval $[-1, 1]$. If it satisfies the hypotheses, compute all numbers c that satisfy the conclusion of the Mean Value Theorem.

$h(x) = \frac{1}{2} \cdot 3^{x+1}$

1. $h(x)$ is an exponential function and thus continuous on the closed interval $[-1,1]$

2. $h'(x) = \frac{1}{2} \cdot 3^{x+1} \cdot \ln(3)$ is also an exponential function and is continuous on $[-1,1]$, thus $h(x)$ is differentiable on the open interval $(-1,1)$.

The Mean Value Theorem: there exists a real number c such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$\frac{1}{2} \cdot 3^{x+1} \cdot \ln(3) = \frac{\frac{1}{2} \cdot 3^2 - \frac{1}{2} \cdot 3^0}{1-(-1)}$$

$$\frac{\ln(3)}{2} \cdot 3^{x+1} = \frac{9/2 - 1/2}{2}$$

$$\frac{\ln(3)}{2} \cdot 3^{x+1} = 2$$

$$3^{x+1} = \frac{4}{\ln(3)}$$

$$(x+1)\ln(3) = \ln\left(\frac{4}{\ln(3)}\right)$$

$$c = \frac{\ln(4/\ln(3)) - \ln(3)}{\ln(3)}$$

4. A car moves in a straight line. At time, t, (measured in seconds), its position (measured in feet) is $s(t) = t^3 - 3t + 2$, for $t \in [0, 4]$. At what time is the instantaneous velocity of the car equal to its average velocity?

$s(t) = t^3 - 3t + 2$

$v(t) = 3t^2 - 3$

$$\bar{v}(t) = \frac{[(4)^3 - 3(4) + 2] - [(0)^3 - 3(0) + 2]}{4 - 0} = \frac{[54] - [2]}{4} = \frac{52}{4} = 13$$

$3t^2 - 3 = 13$

$t^2 = \frac{16}{3}$

$|t| = \sqrt{\frac{16}{3}}$

$t = \sqrt{\frac{16}{3}}$ is in $(0, 4)$

4.3 How Derivatives Affect the Shape of a Graph

5. Determine the intervals on which $f(x) = x^3 - 5x^2 + 8x + 2$ is increasing or decreasing.

$f(x) = x^3 - 5x^2 + 8x + 2$ Domain: $(-\infty, \infty)$

$\frac{df(x)}{dx} = 3x^2 - 10x + 8$ Domain: $(-\infty, \infty)$

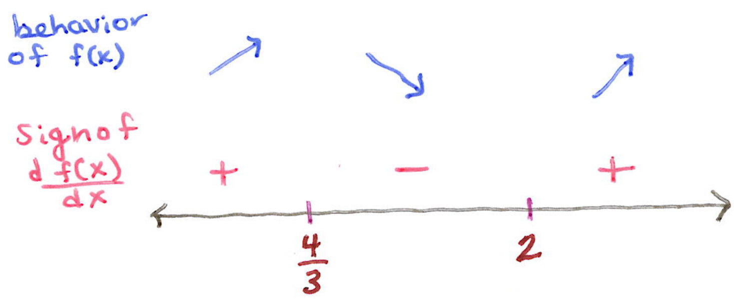
Critical Numbers:

$3x^2 - 10x + 8 = 0$

$(3x-4)(x-2) = 0$

$3x - 4 = 0$ or $x - 2 = 0$

$x = \frac{4}{3}$ $\quad x = 2$



$f(x)$ is increasing on $(-\infty, \frac{4}{3}), (2, \infty)$

$f(x)$ is decreasing on $(\frac{4}{3}, 2)$

6. Determine the intervals on which $f(x) = xe^{x^2-3x}$ is increasing or decreasing.

$f(x) = x \cdot e^{x^2-3x}$ Domain: $(-\infty, \infty)$

$f'(x) = 1e^{x^2-3x} + (2x-3)e^{x^2-3x} \cdot x$ Domain: $(-\infty, \infty)$

$f'(x) = e^{x^2-3x}[1 + (2x-3)x]$

$f'(x) = e^{x^2-3x}(2x^2 - 3x + 1)$

Critical Numbers:

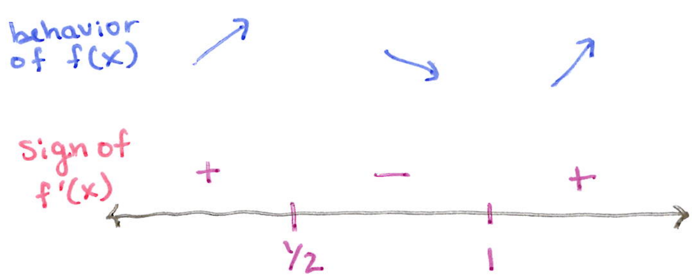
$e^{x^2-3x}(2x^2 - 3x + 1) = 0$

$2x^2 - 3x + 1 = 0$

$(2x-1)(x-1) = 0$

$2x - 1 = 0$ or $x - 1 = 0$

$x = 1/2$ $\quad x = 1$



$f(x)$ is increasing on $(-\infty, 1/2), (1, \infty)$

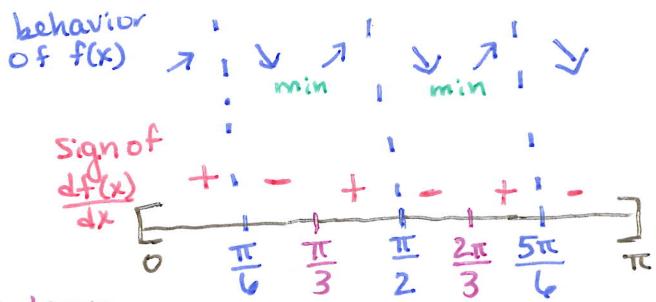
$f(x)$ is decreasing on $(1/2, 1)$

7. Determine the points in the interval $[0, \pi]$ where $f(x) = \sec^2(3x)$ has any local extrema of $f(x)$.

$f(x) = \sec^2(3x)$ Domain: $x \neq \frac{\pi}{6} + \frac{\pi}{3}k$

$\frac{df(x)}{dx} = 2\sec(3x) \cdot (\sec(3x)\tan(3x) \cdot 3)$ Domain: $x \neq \frac{\pi}{6} + \frac{\pi}{3}k$

$\frac{df(x)}{dx} = 6\sec^2(3x)\tan(3x)$



Critical Numbers:

$6\sec^2(3x)\tan(3x) = 0$

$\sec^2(3x) = 0$ or $\tan(3x) = 0$

$\sec(\theta) \neq 0$

$3x = \arctan(0) + \pi k$

$3x = 0 + \pi k$ k is any integer

$x = \frac{\pi}{3}k$

$x = \frac{\pi}{3}, \frac{2\pi}{3}$ 0 & π is not in $(0, \pi)$

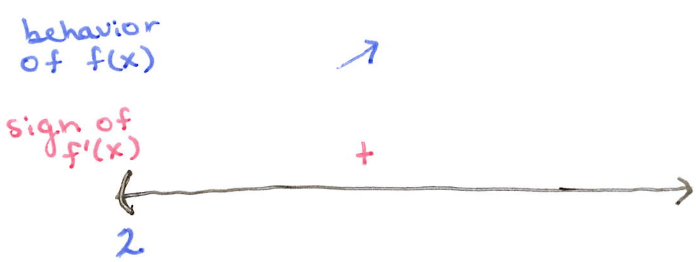
- No local maximums
- Local minimums of $\left(\frac{\pi}{3}, 1\right)$ and $\left(\frac{2\pi}{3}, 1\right)$

8. Determine the points where $f(x) = 5x\ln(3x-6)$ has any local extrema of $f(x)$.

$f(x) = 5x\ln(3x-6)$ Domain: $(2, \infty)$

$f'(x) = 5\ln(3x-6) + \left(\frac{3}{3x-6}\right)5x$ Domain: $(2, \infty)$

$f'(x) = 5\ln(3x-6) + \frac{15x}{3x-6}$



Critical Numbers:

$5\ln(3x-6) + \frac{15x}{3x-6} = 0$

$5\ln(3x-6) = \frac{-15x}{3x-6}$

$\ln(3x-6) \neq \frac{-3x}{3x-6}$ (Check graphically)

No critical numbers

- No local extrema.

9. Determine the intervals where $f(x) = \dfrac{x^2+1}{x}$ is concave up and concave down. Then determine any inflection points of $f(x)$.

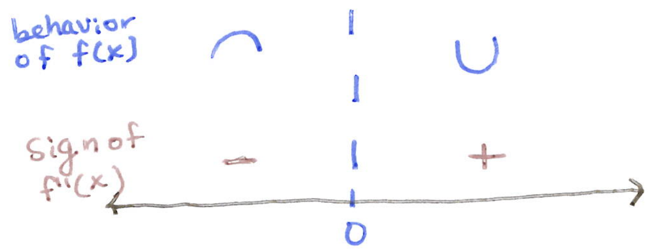
$f(x) = \dfrac{x^2+1}{x}$ Domain: $(-\infty, 0) \cup (0, \infty)$

$f'(x) = \dfrac{2x(x) - 1(x^2+1)}{x^2}$ Domain: $(-\infty, 0) \cup (0, \infty)$

$f'(x) = \dfrac{x^2-1}{x^2}$

$f''(x) = \dfrac{2x(x^2) - 2x(x^2-1)}{x^4}$ Domain: $(-\infty, 0) \cup (0, \infty)$

$= \dfrac{2}{x^3}$



$f''(x) = 0$

$\dfrac{2}{x^3} = 0$

$2 \neq 0$

- $f(x)$ is concave up on $(0, \infty)$
- $f(x)$ is concave down on $(-\infty, 0)$

10. Determine the intervals on which $f(x) = (x^2-16)^{2/3}$ is concave up and concave down. Then determine any inflection points of $f(x)$.

$f(x) = (x^2-16)^{2/3} = \sqrt[3]{(x^2-16)^2}$ Domain: $(-\infty, \infty)$

$f'(x) = \frac{2}{3}(x^2-16)^{-1/3} \cdot (2x) = \frac{4x}{3}(x^2-16)^{-1/3}$ Domain: $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$

$f''(x) = \frac{4}{3}(x^2-16)^{-1/3} + \left[-\frac{1}{3}(x^2-16)^{-4/3} \cdot 2x\right] \cdot \frac{4x}{3}$ Domain: $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$

$= \frac{4}{3}(x^2-16)^{-4/3}\left[(x^2-16) - \frac{2x^2}{3}\right]$

$f''(x)$ does not exist

$\frac{4}{3}(x^2-16)^{4/3} \neq 0$

$x^2 - 16 \neq 0$

$x^2 \neq 16$

$|x| \neq 4$

$x \neq \pm 4$

$f''(x) = 0$

$x^2 - 16 - \frac{2x^2}{3} = 0$

$\frac{1}{3}x^2 = 16$

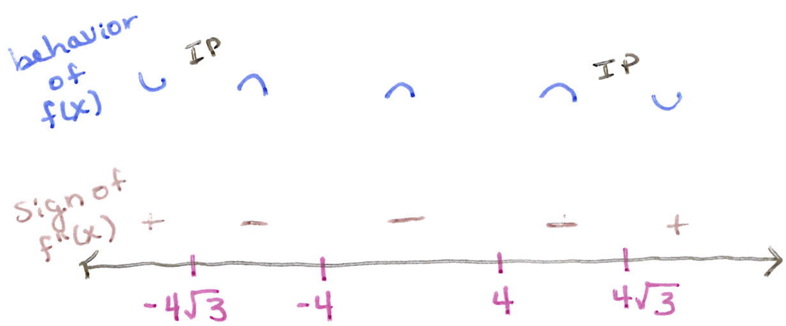
$x^2 = 48$

$|x| = \sqrt{48}$

$x = \pm\sqrt{48}$

or

$\pm 4\sqrt{3}$



- $f(x)$ is concave up on $(-\infty, -4\sqrt{3}), (4\sqrt{3}, \infty)$
- $f(x)$ is concave down on $(-4\sqrt{3}, -4), (-4, 4), (4, 4\sqrt{3})$
- $f(x)$ has inflection points at $(-4\sqrt{3}, (32)^{2/3})$, $(4\sqrt{3}, (32)^{2/3})$

11. Suppose the function $f(x)$ has a domain of $(-\infty, 4) \cup (4, \infty)$, critical numbers of $x = 0$ and $x = 8$, and a second derivatives of $\frac{d^2 f(x)}{dx^2} = \frac{160}{(x-4)^3}$.

Second Derivative Test:

$\left.\frac{d^2 f(x)}{dx^2}\right|_{x=0} = \frac{160}{(0-4)^3} = -$ so local max at $x = 0$

$\left.\frac{d^2 f(x)}{dx^2}\right|_{x=8} = \frac{16}{(8-4)^3} = +$ so local min at $x = 8$

$\frac{d^2 f(x)}{dx^2} = 0$

$\frac{16}{(x-4)^3} = 0$

$16 \neq 0$

None

behavior of $f(x)$: $\cap$ | $\cup$

sign of $\frac{d^2 f(x)}{dx^2}$: $-$ | $+$

4

(a) State the value(s) of x where $f(x)$ has a local maximum: $x = 0$

(b) State the value(s) of x where $f(x)$ has a local minimum: $x = 8$

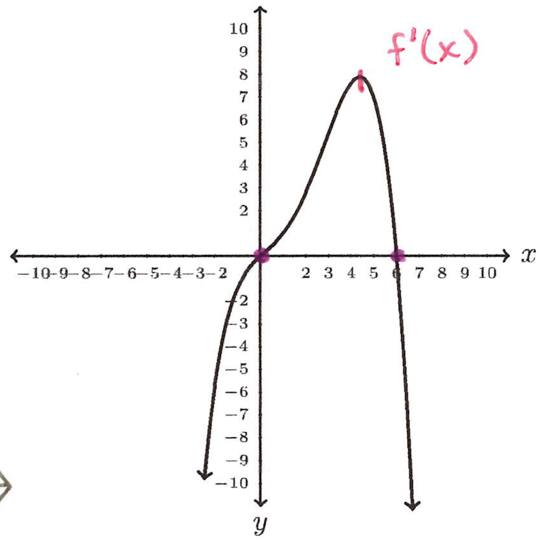
(c) State the "critical numbers" of the second derivative: none

(d) State the interval(s) where $f(x)$ is concave upward: $(4, \infty)$

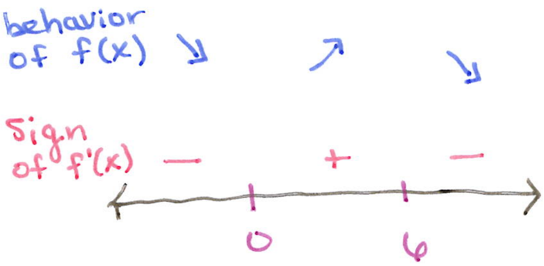
(e) State the interval(s) where $f(x)$ is concave downward: $(-\infty, 4)$

(f) State the value(s) of x where $f(x)$ has an inflection point: none

12. Given the graph of $f'(x)$ below, determine the interval(s) where $f(x)$ is increasing and decreasing.



critical numbers:
$x = 0, 6$

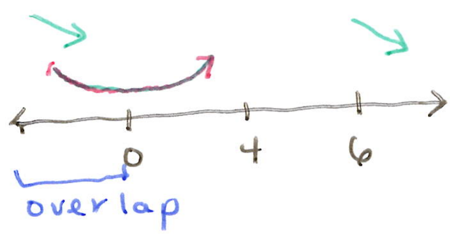


- $f(x)$ is increasing on $(0,6)$
- $f(x)$ is decreasing on $(-\infty,0), (6,\infty)$

13. Using the graph above, determine if there are any intervals where $f(x)$ is decreasing and concave up.

$f(x)$ is concave up where $f'(x)$ is increasing as $f''(x)$ is positive when $f'(x)$ is increasing.

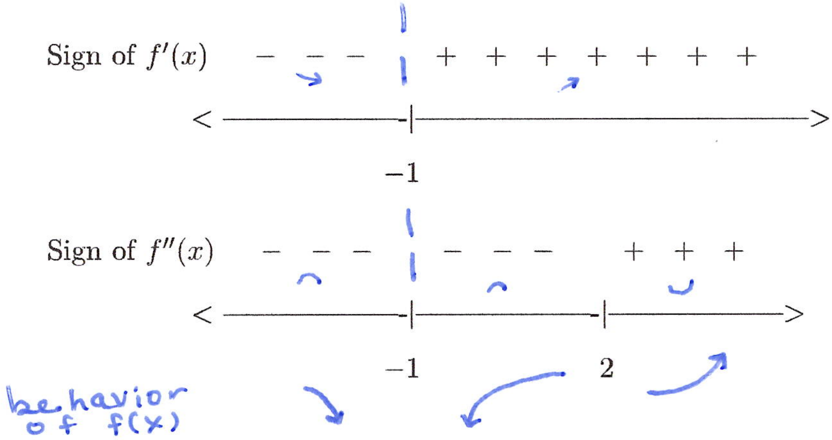
$f'(x)$ is increasing on $(-\infty, 4)$
$f(x)$ is concave up on $(-\infty, 4)$
$f(x)$ is decreasing on $(-\infty, 0) \cup (6, \infty)$



$f(x)$ is decreasing and concave up on $(-\infty, 0)$

14. Given $f(x)$ is a differentiable function on the intervals $(-\infty, -1) \cup (-1, \infty)$, use the sign charts below to answer each question.

We will assume $f(x)$ is continuous on $(-\infty,-1)\cup(-1,\infty)$ because we were not told otherwise



(a) State the critical numbers of the first derivative: none

(b) State the interval(s) where $f(x)$ is increasing: $(-1, \infty)$

(c) State the interval(s) where $f(x)$ is decreasing: $(-\infty, -1)$

(d) State the value(s) of x where $f(x)$ has a local maximum: none

(e) State the value(s) of x where $f(x)$ has a local minimum: none

(f) State the "critical numbers" of the second derivative: $x = 2$

(g) State the interval(s) where $f(x)$ is concave upward: $(2, \infty)$

(h) State the interval(s) where $f(x)$ is concave downward: $(-\infty, -1), (-1, 2)$

(i) State the value(s) of x where $f(x)$ has an inflection point: $x = 2$

15. Sketch the graph of a function that meets these conditions.

$f(x)$ has a domain of all real numbers

$f(4) = 3$ point on graph

$\lim_{x\to\infty} f(x) = -\frac{1}{2}$ horizontal asymptote

$f'(4) = 0$

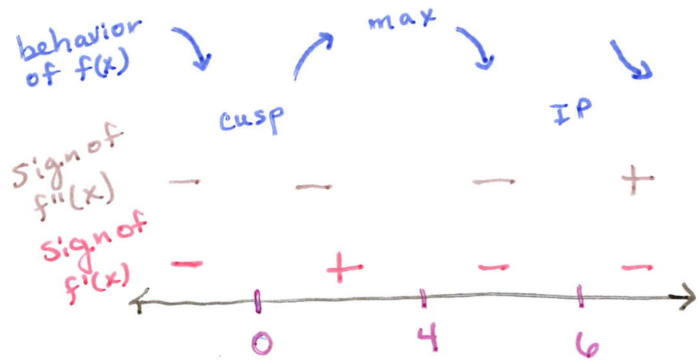
$f'(x)$ does not exist when $x = 0$ and $x = 6$

$f'(x) > 0$ on $(0, 4)$ +

$f'(x) < 0$ on $(-\infty, 0)$, $(4, 6)$, and $(6, \infty)$ −

$f''(x) < 0$ on $(-\infty, 6)$ −

$f''(x) > 0$ on $(6, \infty)$ +



one possible graph