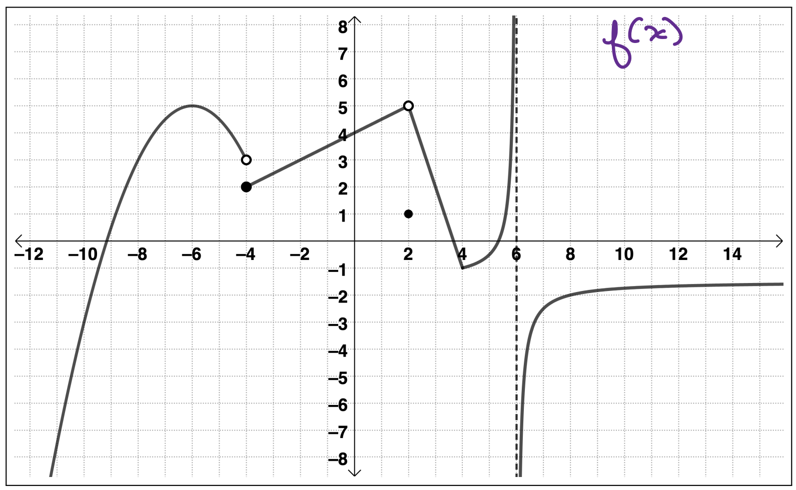
limit as x approaches c of f(x)

Session 1: Sections 1-1 and 1-2

y-value

The notation $\lim_{x \to c} f(x)$ (for a real number c) means we need to find the value $f(x)$ approaches when x is near, *but not necessarily equal to* c. The function must approach the same value (i.e., L) from both the left and right side of $x = c$ for $\lim_{x \to c} f(x) = L$

1. A graph of $f(x)$ is given below. Use the graph to find each limit below. If a limit does not exist, state so and use limit notation to describe any infinite behavior.



from the left

(a) $\lim_{x \to -4^-} f(x)$ = 3

from the right

(b) $\lim_{x \to -4^+} f(x)$ = 2

(c) $\lim_{x \to -4} f(x)$ DNE (Does not exist)

(d) $\lim_{x \to 2} f(x)$ = 5

$\lim_{x \to 2^-} f(x) = 5$ $\lim_{x \to 2^+} f(x) = 5$

from the right

(e) $\lim_{x \to 6^+} f(x) \to -\infty$

(f) $\lim_{x \to 6} f(x)$ DNE

$\lim_{x \to 6^-} f(x) \to \infty$

2. Given $f(x) = \dfrac{5(x+3)}{x^2+5x+6}$, complete the tables below and then use the table to estimate the given limits

(a) $\lim_{x \to -3} f(x)$

left-hand limit		right-hand limit	
x	$f(x)$	x	$f(x)$
-3.1	-4.545	-2.9	-5.556
-3.01	-4.95	-2.99	-5.051
-3.001	-4.995	-2.999	-5.005
-3.0001	-5	-2.9999	-5.001

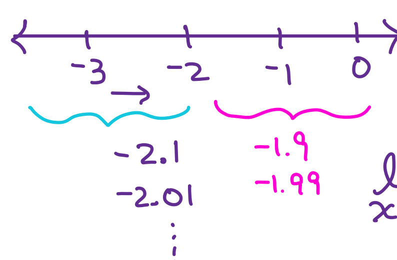
$\lim_{x \to -3^-} f(x) \approx -5$ $\lim_{x \to -3^+} f(x) \approx -5$

$$\boxed{\lim_{x \to -3} f(x) \approx -5}$$

(b) $\lim_{x \to -2} f(x)$

left-hand limit		right-hand limit	
x	$f(x)$	x	$f(x)$
-2.1	-50	-1.9	50
-2.01	-500	-1.99	500
-2.001	-5000	-1.999	5000
-2.0001	-50000	-1.9999	50000

$f(x) = \dfrac{5(x+3)}{x^2+5x+6}$



$\lim_{x \to -2^-} f(x) \to -\infty$ $\lim_{x \to -2^+} f(x) \to \infty$

$$\boxed{\lim_{x \to -2} f(x) \text{ DNE}}$$

Direct Substitution Property For Polynomial and Rational Functions

If P and Q are polynomials and c is any real number, then

$$\lim_{x\to c} P(x) = P(c) \qquad \text{and} \qquad \lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

as long as $Q(c)$ is nonzero.

Cases for a Ratio of Two Functions

Given two functions $f(x)$ and $g(x)$, and any real number c, use the cases below when finding $\lim_{x\to c} \frac{f(x)}{g(x)}$.

- **Case 1:** (L is any real number and $M \neq 0$)

  If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$.

- **Case 2**

  If $\lim_{x\to c} f(x) \neq 0$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)}$ **does not exist**.

- **Case 3**

  If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)}$ cannot be determined (i.e., is indeterminate) and further algebraic manipulation is necessary to convert the limit to an expression in which Case 1 or Case 2 applies.

3. Given $h(x)$ below, find the following limits algebraically, check your results graphically.

$$h(x) = \begin{cases} 2x + 10 & x \leq -7 \\ \dfrac{3x-8}{4-x} & -7 < \underbrace{x < 1}_{\text{left } x=1} \\ \dfrac{2x^2+x-3}{x^2-5x+4} & \underbrace{x \geq 1}_{\text{right } x=1} \end{cases}$$

(a) $\lim_{x\to 1} h(x) = \boxed{-\frac{5}{3}}$

$$\lim_{x\to 1^-} \frac{3x-8}{4-x} = \frac{3(1)-8}{4-1} = \frac{-5}{3}$$

Because $4-1=3 \neq 0$ use direct substitution.

$$\lim_{x\to 1^+} \frac{2x^2+x-3}{x^2-5x+4} = \lim_{x\to 1^+} \frac{(2x+3)\cancel{(x-1)}}{(x-4)\cancel{(x-1)}} = \lim_{x\to 1^+} \frac{2x+3}{x-4} = \frac{2(1)+3}{1-4} = \frac{5}{-3}$$

$$\lim_{x\to 1^+} x^2-5x+4 = 0 \qquad \boxed{\text{CASE 3}}$$

$$\lim_{x\to 1^+} 2x^2+x-3 = 2(1)^2+1-3 = 2+1-3 = 0$$

$$h(x) = \begin{cases} 2x+10 & x \le -7 \\ \dfrac{3x-8}{4-x} & -7 < x < 1 \\ \dfrac{2x^2+x-3}{x^2-5x+4} & x \ge 1 \end{cases}$$

x	$\frac{3x-8}{4-x}$
3.9	37
3.99	397
3.999	3997
4.1	-43
4.01	-403
4.001	-4003

left $\to \infty$

right $\to -\infty$

(b) $\lim_{x\to 4} h(x) = \lim_{x\to 4} \frac{3x-8}{4-x}$ DNE

$\lim_{x\to 4} (3x-8) = 3(4)-8 = 12-8 = 4$

$\lim_{x\to 4} (4-x) = 4-4 = 0$

CASE 2

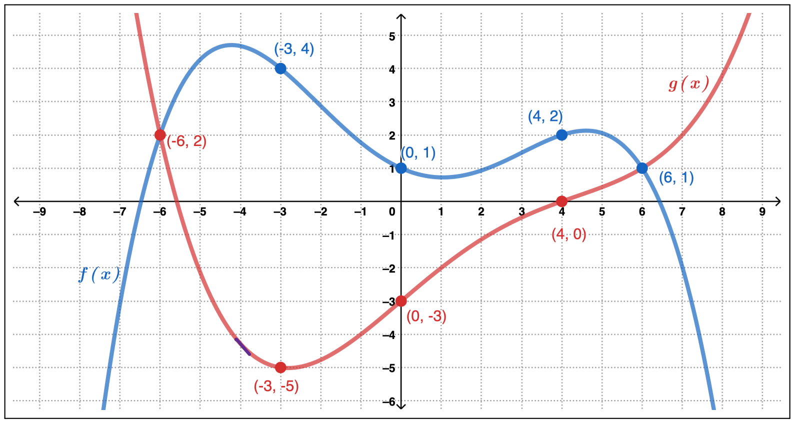
but $\lim_{x\to 4^-} \frac{3x-8}{4-x} \to \infty$ and $\lim_{x\to 4^+} \frac{3x-8}{4-x} \to -\infty$

(c) $\lim_{x\to -7^-} h(x) = \lim_{x\to -7^-} (2x+10) = 2(-7)+10 = -14+10 = \boxed{-4}$

(d) $\lim_{x\to 0} h(x) = \lim_{x\to 0} \frac{3x-8}{4-x} = \frac{3(0)-8}{4-0} = \frac{-8}{4} = \boxed{-2}$

limit laws

4. Given the graph of $f(x)$ and $g(x)$ below find $\lim_{x\to-3}\left(2f(x)+\frac{g(x)}{x^2}+8\right)$.



$$\lim_{x\to-3}\left(2f(x)+\frac{g(x)}{x^2}+8\right) = \lim_{x\to-3} 2f(x) + \lim_{x\to3}\frac{g(x)}{x^2} + \lim_{x\to3} 8$$

$$= 2\lim_{x\to-3} f(x) + \frac{\lim_{x\to3} g(x)}{\left(\lim_{x\to3} x\right)^2} + \lim_{x\to3} 8$$

$$= 2(4) + \frac{-5}{(-3)^2} + 8 = 8 + \frac{-5}{9} + 8 = 16 - \frac{5}{9}$$

$$= 15\frac{4}{9}$$

5. Find the limits below algebraically.

(a) $\lim_{x\to-5}\left[\ln(6+x)-2x\right] = \ln(6+-5) - 2(-5)$

$= \ln(1) + 10$

$= 0 + 10$

$= 10$

(b) $\lim_{x\to4}\frac{x^2-8}{x+4} = \frac{4^2-8}{4+4} = \frac{16-8}{8} = \frac{8}{8} = 1$

(c) $\lim_{x\to 4} \frac{x-4}{x+4} = \frac{4-4}{4+4} = \frac{0}{8} = 0$

(d) $\lim_{x\to 4} \frac{x+4}{x-4}$ DNE, because $\lim_{x\to 4^-} \frac{x+4}{x-4} \to -\infty$, $\lim_{x\to 4^+} \frac{x+4}{x-4} \to \infty$.

$$\left.\begin{array}{l} \lim_{x\to 4} (x+4) = 4+4 = 8 \\ \lim_{x\to 4} (x-4) = 4-4 = 0 \end{array}\right\} \text{CASE 2}$$

X	Y1	
3.9	-79	left
3.99	-799	$\to -\infty$
3.999	-7999	
4.1	81	right
4.01	801	$\to \infty$
4.001	8001	

X=

(e) $\lim_{x\to 1^-} \dfrac{\frac{8}{x+5} - \frac{4}{x+2}}{x-1}$

$$\lim_{x\to 1^-}\left(\frac{8}{x+5} - \frac{4}{x+2}\right) = \frac{8}{1+5} - \frac{4}{1+2} = \frac{8}{6} - \frac{4}{3} = 0$$

CASE 3

$$\lim_{x\to 1^-}(x-1) = 1-1=0$$

$$\lim_{x\to 1^-} \frac{\frac{8}{(x+5)}\frac{(x+2)}{(x+2)} - \frac{4}{(x+2)}\frac{(x+5)}{(x+5)}}{\frac{(x-1)}{1}} = \lim_{x\to 1^-}\left(\frac{8x+16-4x-20}{(x+5)(x+2)} \cdot \frac{1}{(x-1)}\right)$$

$$= \lim_{x\to 1^-} \frac{4x-4}{(x+5)(x+2)(x-1)}$$

$$= \lim_{x\to 1^-} \frac{4\cancel{(x-1)}}{(x+5)(x+2)\cancel{(x-1)}}$$

$$= \lim_{x\to 1^-} \frac{4}{(x+5)(x+2)} = \frac{4}{(1+5)(1+2)}$$

$$= \frac{4}{6\cdot 3} = \frac{4}{18} = \boxed{\frac{2}{9}}$$

(f) $\lim_{x\to -1/2} f(x)$ given $f(x) = \begin{cases} \dfrac{2x^2-3x-2}{2x+1} & x < -\frac{1}{2} \\ 2x+7 & x > -\frac{1}{2} \end{cases}$ $\left.\right\}$ left of $x = -\frac{1}{2}$ $\left.\right\}$ right of $x = -\frac{1}{2}$

$$\lim_{x\to -\frac{1}{2}^-} \frac{2x^2-3x-2}{2x+1} = \lim_{x\to -\frac{1}{2}^-} \frac{\cancel{(2x+1)}(x-2)}{\cancel{(2x+1)}} = \lim_{x\to -\frac{1}{2}^-}(x-2) = -\frac{1}{2} - 2 = -2.5$$

$$2\left(-\tfrac{1}{2}\right)+1 = -1+1=0$$

$$2\left(-\tfrac{1}{2}\right)^2 - 3\left(-\tfrac{1}{2}\right) - 2 = 2\left(\tfrac{1}{4}\right) + \tfrac{3}{2} - 2$$

$$= \tfrac{1}{2} + \tfrac{3}{2} - 2$$

$$= 0$$

CASE 3

$$\lim_{x\to -\frac{1}{2}^+}(2x+7) = 2\left(-\tfrac{1}{2}\right)+7 = -1+7=6$$

$$\boxed{\lim_{x\to -\frac{1}{2}} f(x) \text{ DNE}}$$