Math 151 - Week-In-Review 4

Topics for the week:

2.7 Derivatives and Rates of Change
2.8 The Derivative as a Function
J.1 through 2.8 Exam Review

2.7 Derivatives and Rates of Change

1. Write the equation of the line tangent to the graph of $h(x) = 5x - x^2$ at the point $(-1, -6)$.

Want $y = f'(a)(x-a) + f(a)$

$a = -1$

$h(a) = 5(-1) - (-1)^2$
$= -5 - 1$
$h(-1) = -6$

Need $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

so

$$h'(-1) = \lim_{x \to -1} \frac{h(x) - h(-1)}{x - (-1)}$$

$$= \lim_{x \to -1} \left(\frac{(5x - x^2) - (-6)}{x + 1} \right)$$

$$= \lim_{x \to -1} \left(\frac{-x^2 + 5x + 6}{x + 1} \right)$$

$$= \lim_{x \to -1} \left(\frac{-(x-6)(x+1)}{x+1} \right)$$

$$= \lim_{x \to -1} \left(-(x-6) \right)$$

$$= -(-7)$$

$h'(-1) = 7$

$y = 7(x - (-1)) + (-6)$ or $y = 7x + 1$

2. The displacement, in feet, of a particle moving in a straight line is given by $y = \sqrt{10-3t}$, where t is measured in minutes.

(a) Compute the average velocity over the interval [2, 3].

Average Velocity $= m_{sec} = \dfrac{f(b)-f(a)}{b-a}$

$$m_{sec} = \frac{y(3)-y(2)}{3-2}$$

$$= \frac{\sqrt{10-3(3)}-\sqrt{10-3(2)}}{1}$$

$$= \frac{1-2}{1}$$

$$m_{sec} = -1$$

(b) Compute the instantaneous velocity when $t = 2$.

Instantaneous Velocity $= m_{tan} = \lim\limits_{t\to a}\left[\dfrac{f(t)-f(a)}{t-a}\right]$

$$m_{tan} = \lim_{t\to 2}\left[\frac{\sqrt{10-3t}-\sqrt{10-3(2)}}{t-2}\right]$$

$$= \lim_{t\to 2}\left[\frac{\sqrt{10-3t}-2}{t-2}\right]$$

$$= \lim_{t\to 2}\left[\left(\frac{\sqrt{10-3t}-2}{t-2}\right)\left(\frac{\sqrt{10-3t}+2}{\sqrt{10-3t}+2}\right)\right]$$

we multiply by a "1" to eliminate the square roots in the numerator.

$$= \lim_{t\to 2}\left[\frac{(10-3t)-4}{(t-2)(\sqrt{10-3t}+2)}\right]$$

$$= \lim_{t\to 2}\left[\frac{-3t+6}{(t-2)\sqrt{10-3t}+2}\right]$$

$$= \lim_{t\to 2}\left[\frac{-3}{\sqrt{10-3t}+2}\right] = -\frac{3}{4} = m_{tan}$$

2.8 The Derivative as a Function

3. Compute the derivative of the function $g(x) = \frac{x}{x-4}$, using the definition of the derivative. Then state the domain of both $g(x)$ and $g'(x)$.

$g(x) = \frac{x}{x-4}$ Domain of $g(x)$: $x \in (-\infty, 4) \cup (4, \infty)$

$$g'(x) = \lim_{h \to 0} \left[\frac{g(x+h) - g(x)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\frac{x+h}{x+h-4} - \frac{x}{x-4}}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\left(\frac{x+h}{x+h-4} - \frac{x}{x-4} \right)}{h} \cdot \frac{(x+h-4)(x-4)}{(x+h-4)(x-4)} \right]$$

Multiply by the common denominator

$$= \lim_{h \to 0} \left[\frac{(x+h)(x-4) - x(x+h-4)}{h(x+h-4)(x-4)} \right]$$

$$= \lim_{h \to 0} \left[\frac{x^2 - 4x - 4h + xh - x^2 - xh + 4x}{h(x+h-4)(x-4)} \right]$$

$$= \lim_{h \to 0} \left[\frac{-4h}{h(x+h-4)(x-4)} \right]$$

$$= \lim_{h \to 0} \left[\frac{-4}{(x+h-4)(x-4)} \right]$$

$$g'(x) = \frac{-4}{(x-4)(x-4)} = \frac{-4}{(x-4)^2}$$

Domain of $g(x)$: $x \in (-\infty, 4) \cup (4, \infty)$

4. Given $y = 4x^2 - 11x + 25$,

(a) show $\frac{dy}{dx} = 8x - 11$.

$$\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{(4(x+h)^2 - 11(x+h) + 25) - (4x^2 - 11x + 25)}{h} \right]$$

Note:
$(x+h)^2 = x^2 + 2xh + h^2$

$$= \lim_{h \to 0} \left[\frac{4(x^2+2xh+h^2) - 11x - 11h + 25 - 4x^2 + 11x - 25}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{4x^2 + 8xh + 4h^2 - 11x - 11h \boxed{+25} - 4x^2 + 11x \boxed{-25}}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{8xh + 4h^2 - 11h}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{h(8x + 4h - 11)}{h} \right]$$

$$= \lim_{h \to 0} (8x + 4h - 11)$$

$$\frac{dy}{dx} = 8x - 11 \checkmark$$

(b) Is there any point along the curve where the tangent line is horizontal?

Need $8x - 11 = 0 \leftarrow m_{tan} = 0$

$8x - 11 = 0$

$8x = 11$

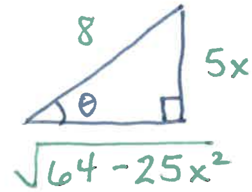
$x = 11/8$ so yes.

Exam Review (J.1 - 2.8)

6. Simplify the expression $\tan\left(\arcsin\left(\frac{5x}{8}\right)\right)$

$\theta = \arcsin\left(\frac{5x}{8}\right)$

$\sin(\theta) = \frac{5x}{8}$



$\tan(\theta) = \frac{5x}{\sqrt{64-25x^2}}$

$-\frac{8}{5} \le x \le \frac{8}{5}$

7. Given the points $J(0,5)$ and $K(-2,0)$, compute a vector of length $\frac{1}{2}$ that is in the same direction as $\overrightarrow{JK}$.

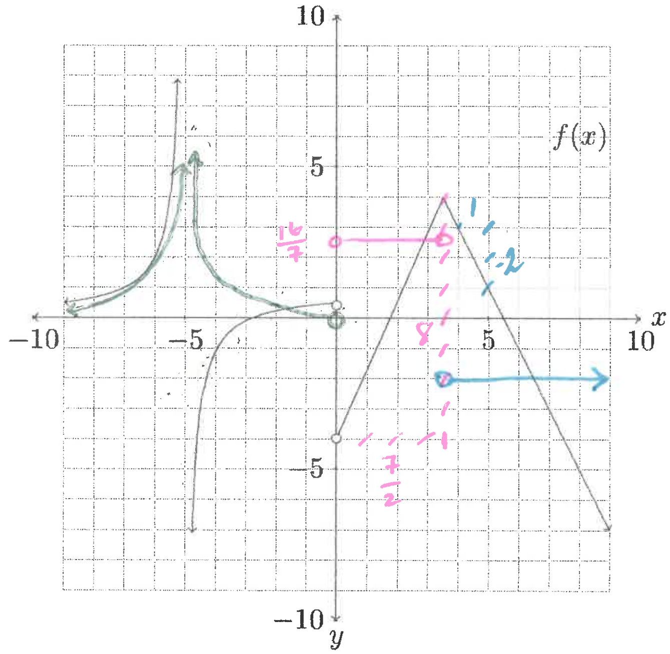
$\vec{v} = \langle -2-0,\ 0-5 \rangle = \langle -2,-5 \rangle$

$\|\vec{v}\| = \sqrt{(-2)^2+(-5)^2}$
$= \sqrt{4+25}$
$= \sqrt{29}$

$\vec{w} = \frac{1}{2} \cdot \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{2}\left\langle \frac{-2}{\sqrt{29}}, \frac{-5}{\sqrt{29}} \right\rangle$

$= \left\langle \frac{-1}{\sqrt{29}}, \frac{-5}{2\sqrt{29}} \right\rangle$

5. Given the graph of $f(x)$ below,

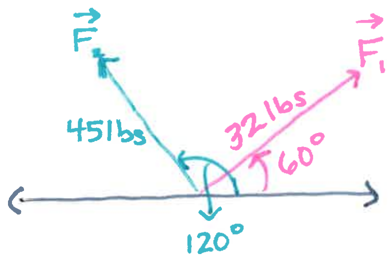


(a) Sketch the graph of $f'(x)$ using $f(x)$.

(b) State the values of x at which $f(x)$ is not differentiable.

at $x = -5, 0, 3.5$

8. Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on an object. The force $\mathbf{F}_1$ has a magnitude of 32 lbs and a direction of $60°$ counterclockwise from the positive x-axis, and $\mathbf{F}_2$ has a magnitude of 45 lbs and a direction of $120°$ counterclockwise from the positive x-axis. State the resultant force $\mathbf{F}$.



$\vec{F}_1 = \langle 32\cos(60°), 32\sin(60°) \rangle$

$= \langle 16, 16\sqrt{3} \rangle$

$\vec{F}_2 = \langle 45\cos(120°), 45\sin(120°) \rangle$

$= \langle -\frac{45}{2}, \frac{45\sqrt{3}}{2} \rangle$

$\vec{F} = \vec{F}_1 + \vec{F}_2 = \langle 16 + (-\frac{45}{2}), 16\sqrt{3} + \frac{45\sqrt{3}}{2} \rangle$

$= \langle 16 - \frac{45}{2}, 16\sqrt{3} + \frac{45\sqrt{3}}{2} \rangle$

9. A force is given by a vector $\mathbf{F} = 2\mathbf{i} + 5\mathbf{j}$ and moves an object from the point $M(4,2)$ to the point $N(7,6)$. Compute the work done.

$W = \vec{F} \cdot \vec{D}$

$= \langle 2,5 \rangle \cdot \langle 3,4 \rangle$

$= 2 \cdot 3 + 5 \cdot 4$

$W = 26$

$\vec{D} = \langle 7-4, 6-2 \rangle$

$= \langle 3, 4 \rangle$

$\vec{F} = \langle 2, 5 \rangle$

10. Compute the angle between the vectors $\mathbf{a} = \langle -2, 9\rangle$ and $\mathbf{b} = \langle 8, 4\rangle$.

$$\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|}$$

$$\cos(\theta) = \frac{-2\cdot 8 + 9\cdot 4}{(\sqrt{4+81})(\sqrt{64+16})}$$

$$= \frac{20}{\sqrt{85}\cdot\sqrt{80}}$$

$$= \frac{20}{20\sqrt{17}}$$

$$\cos(\theta) = \frac{1}{\sqrt{17}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{17}}\right) \qquad 0 < \theta < \frac{\pi}{2}$$

11. Write a parametric equation of the line passing through the points $(12, 5)$ and $(9, -2)$.

$$\vec{r}(t) = \vec{r}_0 + \vec{r}\cdot t$$

$$= \langle 12, 5\rangle + \langle 3, 7\rangle t$$

$$\vec{r}(t) = \langle 12+3t, 5+7t\rangle$$

$$x(t) = 12 + 3t$$
$$y(t) = 5 + 7t$$

$$\vec{r}_0 = \langle 12, 5\rangle$$

$$\vec{r} = \langle 12-9, 5-(-2)\rangle$$
$$= \langle 3, 7\rangle$$

12. Determine the parametric equations for the line that passes through the point (3,-1) and is

(a) is parallel to the vector $\langle -5, -4 \rangle$.

$\vec{r}(t) = \langle 3, -1 \rangle + \langle -5, -4 \rangle t$

$= \langle 3 - 5t, -1 - 4t \rangle$

$x(t) = 3 - 5t$
$y(t) = -1 - 4t$

(b) is perpendicular to the vector $\langle -5, -4 \rangle$.

$a_\perp = \langle 4, -5 \rangle$ or $\langle -4, 5 \rangle$

$\vec{r}(t) = \langle 3, -1 \rangle + \langle 4, -5 \rangle t$

$= \langle 3 + 4t, -1 - 5t \rangle$

$x(t) = 3 + 4t$
$y(t) = -1 - 5t$

13. State the slope of the line with corresponding vector equation $\mathbf{r}(t) = \langle 5 - 2t, -8 + 7t \rangle$.

$\vec{r}(t) = \langle 5 - 2t, -8 + 7t \rangle$

$= \langle 5, -8 \rangle + \langle \underset{\Delta x}{-2}, \underset{\Delta y}{7} \rangle t$

$m = -\frac{7}{2}$

14. Determine whether the lines, $L_1 = \mathbf{r}(t) = (-6 + 2t)\mathbf{i} + (7 - 6t)\mathbf{j}$ and $L_2 = \mathbf{r}(s) = \left(5 + \frac{1}{2}s\right)\mathbf{i} + \left(-8 + \frac{3}{2}s\right)\mathbf{j}$, are parallel, perpendicular, or neither.

$L_1 = \vec{r}(t) = \langle -6, 7 \rangle + \langle 2, -6 \rangle t$ $\quad m_t = \frac{-6}{2} = -3$

$L_2 = \vec{r}(s) = \langle 5, -8 \rangle + \langle \frac{1}{2}, \frac{3}{2} \rangle s$ $\quad m_s = \frac{3/2}{1/2} = 3$

$m_t \neq m_s$ so L_1 and L_2 are not parallel

$m_t \cdot m_s \neq -1$ so L_1 and L_2 are also not perpendicular

15. Evaluate each limit:

(a) $\lim_{x\to 6^+} \left(\frac{\overbrace{x+1}^{\to 7}}{\underbrace{x-6}_{\to 0^+}}\right) = \infty$

(b) $\lim_{x\to 2\pi^-} (x\csc(x)) = \lim_{x\to 2\pi^-} \left(\frac{\overbrace{x}^{\to 2\pi}}{\underbrace{\sin(x)}_{\to 0^-}}\right)$

$= -\infty$

(c) $\lim_{x\to 5} \left(\frac{5-x}{x^2-25}\right) = \lim_{x\to 5}\left(\frac{-\cancel{(x-5)}}{\cancel{(x-5)}(x+5)}\right)$

$= \lim_{x\to 5}\left(\frac{-1}{\underbrace{x+5}_{\to 10}}\right)$

$= -\frac{1}{10}$

16. Evaluate the limit $\lim\limits_{x \to 7^+} \left(\dfrac{|7-x|}{x-7} \right)$

$$|7-x| = \begin{cases} -(7-x) & \text{if } x > 7 \\ 7-x & \text{if } x \le 7 \end{cases}$$

$$\lim_{x \to 7^+} \left(\frac{-(7-x)}{x-7} \right) = \lim_{x \to 7^+} \left(\frac{x-7}{x-7} \right)$$

$$= \lim_{x \to 7^+} (1)$$

$$= 1$$

17. Show that $f(x) = \begin{cases} \sin(x) & x < \frac{\pi}{2} \\ \frac{x}{2} - \frac{\pi}{4} & x \ge \frac{\pi}{2} \end{cases}$ is not continuous at $x = \frac{\pi}{2}$.

1. $f(\pi/2) = \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{\pi}{4} = 0$

2. $\lim\limits_{x \to \frac{\pi}{2}^-} f(x) = \lim\limits_{x \to \frac{\pi}{2}^-} (\sin(x)) = 1$ or $\lim\limits_{x \to \frac{\pi}{2}^+} f(x) = \lim\limits_{x \to \frac{\pi}{2}^+} \left(\frac{x}{2} - \frac{\pi}{4}\right) = 0$

Since $\lim\limits_{x \to \frac{\pi}{2}^-} f(x) \neq \lim\limits_{x \to \frac{\pi}{2}^+} f(x)$, the $\lim\limits_{x \to \frac{\pi}{2}} f(x)$ does not exist

And thus the function is not continuous.

18. Determine the values of a and b such that $f(x)$ is continuous over the real numbers.

$$f(x) = \begin{cases} ax^2 - 5x + b & \text{if } x < -1 \\ -\frac{65}{2}x - \frac{39}{2} & \text{if } -1 \le x < 1 \\ bx^3 - 9ax & \text{if } x \ge 1 \end{cases}$$

$x=-1$

1. $f(-1) = \frac{65}{2} - \frac{39}{2} = 13$

2. $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} (ax^2 - 5x + b)$
$= a + 5 + b$

$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} \left(-\frac{65}{2}x - \frac{39}{2}\right)$
$= 13$

$\lim\limits_{x\to -1} f(x)$ exists if $a+b+5=13$

$x=1$

1. $f(1) = b - 9a$

2. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} \left(-\frac{65}{2}x - \frac{39}{2}\right)$
$= -52$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (bx^3 - 9ax)$
$= b - 9a$

$\lim\limits_{x\to 1} f(x)$ exists if $-52 = b - 9a$

$a+b+5=13$
$a+b=8$
$b=8-a$

$-52 = b - 9a$
$-52 = -9a + b$

$-52 = -9a + (8-a)$
$-60 = -10a$
$6 = a$

$\boxed{\begin{matrix} a=6 \\ b=2 \end{matrix}}$

19. Evaluate the limit, $\lim\limits_{x\to\infty} \frac{3e^{2x}}{2e^{2x} - e^x}$, if possible.

$$\lim_{x\to\infty} \left(\frac{3e^{2x}}{2e^{2x} - e^x}\right) = \lim_{x\to\infty} \left[\left(\frac{3e^{2x}}{2e^{2x} - e^x}\right)\left(\frac{\frac{1}{e^{2x}}}{\frac{1}{e^{2x}}}\right)\right]$$

$$= \lim_{x\to\infty} \left[\frac{\frac{3e^{2x}}{e^{2x}}}{\frac{2e^{2x}}{e^{2x}} - \frac{e^x}{e^{2x}}}\right]$$

$$= \lim_{x\to\infty} \left[\frac{3}{2 - \underbrace{1/e^x}_{\to 0}}\right]$$

$$= \frac{3}{2}$$

20. Determine the left end-behavior (as $x \to -\infty$) of the function $g(x) = \dfrac{-\sqrt{3x^4+2x^2}+5}{4x^4-x^2}$.

$$\lim_{x\to-\infty}\left[\frac{-\sqrt{3x^4+2x^2}+5}{4x^4-x^2}\right] = \lim_{x\to-\infty}\left[\left(\frac{-\sqrt{3x^4+2x^2}+5}{4x^4-x^2}\right)\cdot\left(\frac{\frac{1}{x^4}}{\frac{1}{x^4}}\right)\right]$$

$$= \lim_{x\to-\infty}\left[\frac{\frac{-\sqrt{3x^4+2x^2}}{x^4}+\frac{5}{x^4}}{\frac{4x^4}{x^4}-\frac{x^2}{x^4}}\right]$$

$$= \lim_{x\to-\infty}\left[\frac{-\sqrt{\frac{3x^4}{x^8}+\frac{2x^2}{x^8}}+5/x^4}{4-1/x^2}\right]$$

$$= 0$$

21. Evaluate the limit, $\lim\limits_{x\to\infty}\left[\ln(3+6x^2)-\ln(x^3+x-1)\right]$, if possible.

$$\lim_{x\to\infty}\left[\ln(3+6x^2)-\ln(x^3+x-1)\right] = \lim_{x\to\infty}\left[\ln\left(\frac{3+6x^2}{x^3+x-1}\right)\right]$$

$$= \ln\left[\lim_{x\to\infty}\left(\frac{6x^2+3}{x^3+x-1}\right)\right]$$

$$= \ln\left[\underbrace{\lim_{x\to\infty}\left(\frac{\overbrace{\frac{6}{x}+3/x^3}^{\to 0}}{\underbrace{1+1/x^2-1/x^3}_{\to 1}}\right)}_{\to 0}\right]$$

$$= -\infty$$