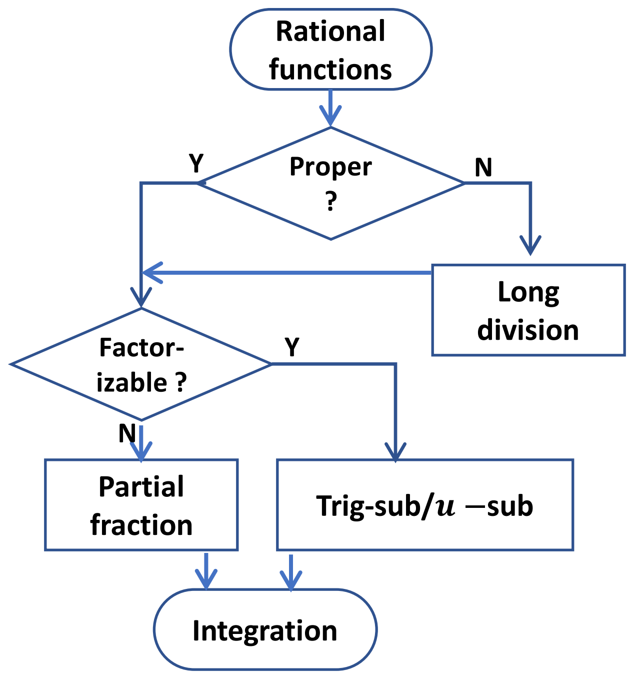
Week in Review
Math 152

Week 06

Integration by Partial Fractions

Improper Integrals

Integration by Partial Fractions



Integration by Partial Fractions

The quotient remainder theorem and general principles of the long division algorithm

$\boldsymbol{x = dQ + R}$

- x:dividend ($\in \mathbb{N}$)
- d: divisor
- Q: quotient
- R: remainder ($x - Qd$)

$$\begin{array}{r|l} & Q \\ \hline d \;) & x \\ & Qd \\ \hline & x - Qd \quad \leftarrow \boldsymbol{R} \end{array}$$

The long division algorithm for polynomials

quotient remainder by orders

$x^2 = (x+1)(x-1) + 1$

$$\begin{array}{r|l} & x \quad -1 \\ \hline x+1\;) & x^2 \\ & x^2 + x \\ \hline & \quad -x \\ & \quad -x - 1 \\ \hline & \qquad 1 \end{array} \qquad \begin{array}{r|l} & 1 \\ \hline x+1\;) & x - 1 \\ & x + 1 \\ \hline & \quad -2 \end{array}$$

Application to improper rational function

$$\frac{x^2}{x+1} = \frac{(x+1)(x-1)+1}{x+1}$$

$$= (x+1) + \frac{1}{(x+1)}$$

long division algorithm for polynomials for Tylor expansion

(*Repeating long divisions for the quotients*)

$(x-1) = (x+1)\cdot 1 - 2$

$$\begin{aligned} x^2 &= (x+1)(x-1) + 1 \\ &= (x+1)\big((x+1)\cdot 1 - 2\big) + 1 \\ &= (x+1)^2 - 2(x+1) + 1 \end{aligned}$$

Integration by Partial Fractions

Evaluate $\displaystyle\int_0^1 \frac{4x^2+5}{2x+1}\,dx$

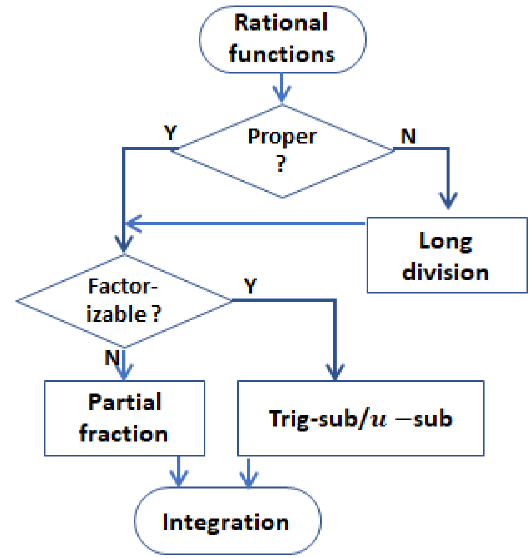
(a) $2\ln 3$

(b) $3\ln 3$

(c) $4\ln 3$

(d) $6\ln 3$

(e) None of these



Long division

$$
\begin{array}{r|lll}
 & 2x & -1 & \\
\hline
2x+1 & 4x^2 & +0x & +5 \\
 & 4x^2 & +2x & \\
\hline
 & & -2x & +5 \\
 & & -2x & -1 \\
\hline
 & & & 6
\end{array}
$$

Proper rational function

$$\int \left(2x-1+\frac{6}{2x+1}\right)dx$$

$$= \int (2x-1)dx + \int \frac{6}{2x+1}dx$$

$$= x^2 - x + 3\ln|2x+1| + C$$

$$\int_0^1 \frac{4x^2+5}{(2x+1)}dx = [x^2 - x + 3\ln|2x+1|]_0^1 = 3\ln 3$$

u-sub :$u = 2x+1$; $dx = \frac{1}{2}du$

$$\int \frac{6}{2x+1}dx = 3\int \frac{1}{u}du = 3\ln|u| + C$$

$$= 3\ln|2x+1| + C$$

Integration by Partial Fractions

Compute $\displaystyle\int_0^4 \frac{x+2}{x^2+4}\,dx.$

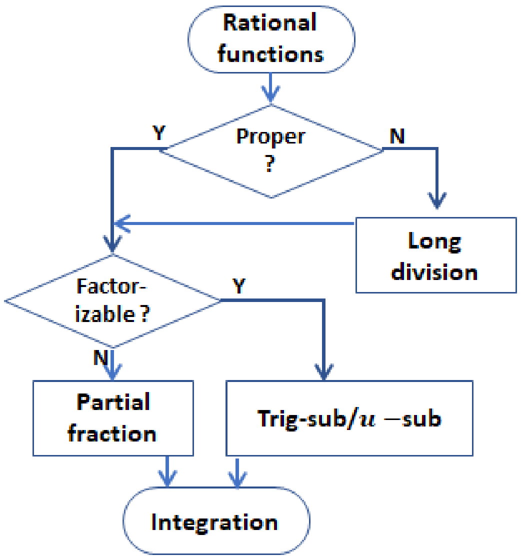
(a) $\frac{1}{2}(\ln 20 - \ln 4) + \arctan(2)$

(b) $\ln 6 - \ln 2$

(c) $\ln 20 - \ln 4$

(d) $\frac{1}{2}(\ln 20 - \ln 4) + 2\arctan(4)$

(e) $\ln 20 - \ln 4 + 2\arctan(4)$



Integration by Partial Fractions

$$\int \frac{x^3 + x}{x - 1}\, dx =$$

(a) $\frac{x^3}{3} + \frac{x^2}{2} + 2x - 2\ln|x-1| + C$

(b) $\frac{x^3}{3} - \frac{x^2}{2} + 2x + 2\ln|x-1| + C$

(c) $\frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$

(d) $\frac{x^3}{3} + \frac{x^2}{2} - 2x + 2\ln|x-1| + C$

(e) $\frac{x^3}{3} - \frac{x^2}{2} + 2x - 2\ln|x-1| + C$

Integration by Partial Fractions

Prerequisite : Factor denominators completely

- $\frac{1}{(x+1)(x^2-2x-2)(x^2-2x+2)} = \frac{1}{(x+1)(x-3)(x+1)(x^2-2x+2)}$

Proper factor rule : Break down to proper rational functions

- $\frac{3x}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{(x-1)}$
- $\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$

Linear factor rule : $\frac{\boldsymbol{p(x)}}{\boldsymbol{(ax+b)^m}} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$ w/ $\deg\big(p(x)\big) < m$

$$\frac{x^2+4x+1}{(x+1)^3} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} - \frac{C}{(x+1)^3}$$

$$\frac{x^2+4x+1}{(x+1)^3} = \frac{(x+1)^2+2(x+1)-1}{(x+1)^3} = \frac{1}{(x+1)} + \frac{2}{(x+1)^2} - \frac{1}{(x+1)^3}$$

Integration by Partial Fractions

Cover-up method

- $$\frac{1}{(x-a)(x-b)} = \left[\frac{1}{(x-a)\,(x-b)}\right]_{x=b} \cdot \frac{1}{(x-b)} + \left[\frac{1}{(x-a)\,(x-b)}\right]_{x=a} \cdot \frac{1}{(x-a)}$$

  Decompose "cover-up" method.

  $$= \frac{1/(b-a)}{x-b} + \frac{1/(a-b)}{x-a}$$

Find the partial fraction decomposition of

- $\frac{1}{(x-1)(x-2)}$

- $\frac{1}{(x-1)(x-2)(x-3)}$

- $\frac{1}{(x-1)(x-2)(x-3)(x-4)}$

Integration by Partial Fractions

Cover-up method w/ non-factorizable denominator factor

$$\frac{3x}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{(x-1)}$$

Step 1: Multiply $(x-1)$ on both sides and let $x=1$

- $\frac{3}{1^2+2} = 1 = C$

Step 2: Pass $(x-1)$ term to the LHS and simplify the LHS

- $\frac{3x}{(x^2+2)(x-1)} - \frac{1}{(x-1)} = \frac{Ax+B}{x^2+2}$

- $\frac{3x-(x^2+2)}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2}$
- $\frac{-(x-1)(x-2)}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2}$
- $\frac{-(x-2)}{(x^2+2)} = \frac{Ax+B}{x^2+2}$

  $A=-1, B=2$

$$\frac{3x}{(x^2+2)(x-1)} = \frac{-x+2}{x^2+2} + \frac{1}{(x-1)}$$

Integration by Partial Fractions

Which of the following is the form of the partial-fraction decomposition for the rational function?

$$\frac{1}{(x+1)(x^2-2x-3)(x^2-2x+2)}$$

(a) $\frac{A}{x+1}+\frac{Bx+C}{x^2-2x-3}+\frac{Dx+E}{x^2-2x+2}$

(b) $\frac{A}{x+1}+\frac{Bx+C}{(x+1)^2}+\frac{D}{x-3}+\frac{Ex+F}{x^2-2x+2}$

(c) $\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-3}+\frac{D}{x^2-2x+2}$

(d) $\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-3}+\frac{Dx+E}{x^2-2x+2}$

(e) $\frac{A}{x+1}+\frac{B}{x^2-2x-3}+\frac{C}{x^2-2x+2}$

Integration by Partial Fractions

Compute the following integral showing all necessary work clearly.

$$\int \frac{4x^2 - 5x + 11}{(x+1)(x-1)(x^2+4)}\, dx$$

Integration by Partial Fractions

Find $\displaystyle\int \frac{x+2}{x^2(x^2+1)}\,dx$

Integration by Partial Fractions (Exercise)

Write out the form of the partial fraction decomposition of the function

$$f(x) = \frac{x^3 - 2x^2 - 5x + 4}{(x+2)^2(x^2-1)(x^2+5x+7)}$$

(a) $\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1} + \frac{D}{x+1} + \frac{Ex+F}{x^2+5x+7}$

(b) $\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2-1} + \frac{Ex+F}{x^2+5x+7}$

(c) $\frac{A}{(x+2)^2} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{Dx+E}{x^2+5x+7}$

(d) $\frac{A}{(x+2)^2} + \frac{B}{x^2-1} + \frac{Cx+D}{x^2+5x+7}$

(e) $\frac{Ax+B}{(x+2)^2} + \frac{Cx+D}{x^2-1} + \frac{Ex+F}{x^2+5x+7}$

$$\int \frac{3-x}{x^2+3x-4}\,dx =$$

(a) $\frac{-7}{5}\ln|x+4| + \frac{2}{5}\ln|x-1| + C$

(b) $\frac{7}{2}\ln|x+4| + \frac{2}{5}\ln|x-1| + C$

(c) $\frac{1}{5}\ln|x-4| - \frac{4}{5}\ln|x+1| + C$

(d) $\frac{-1}{5}\ln|x-4| + \frac{4}{5}\ln|x+1| + C$

(e) $\frac{2}{5}\ln|x+4| - \frac{7}{5}\ln|x-1| + C$

Integration by Partial Fractions (Exercise)

Which of the following is a proper Partial Fraction Decomposition for $\dfrac{x+1}{(x^2-16)(x-3)^2(x^2+1)}$?

(a) $\dfrac{x+1}{(x^2-16)(x-3)^2(x^2+1)} = \dfrac{Ax+B}{x^2-16} + \dfrac{C}{x-3} + \dfrac{D}{(x-3)^2} + \dfrac{Ex+F}{x^2+1}$

(b) $\dfrac{x+1}{(x^2-16)(x-3)^2(x^2+1)} = \dfrac{A}{x-4} + \dfrac{B}{x+4} + \dfrac{C}{x-3} + \dfrac{D}{(x-3)^2} + \dfrac{Ex+F}{x^2+1}$

(c) $\dfrac{x+1}{(x^2-16)(x-3)^2(x^2+1)} = \dfrac{Ax+B}{x^2-16} + \dfrac{Cx+D}{(x-3)^2} + \dfrac{Ex+F}{x^2+1}$

(d) $\dfrac{x+1}{(x^2-16)(x-3)^2(x^2+1)} = \dfrac{A}{x-4} + \dfrac{B}{x+4} + \dfrac{C}{x-3} + \dfrac{D}{(x-3)^2} + \dfrac{E}{x^2+1}$

(e) $\dfrac{x+1}{(x^2-16)(x-3)^2(x^2+1)} = \dfrac{A}{x-4} + \dfrac{B}{(x-4)^2} + \dfrac{C}{x-3} + \dfrac{D}{(x-3)^2} + \dfrac{Ex+F}{x^2+1}$

Which of the following is a proper Partial Fraction Decomposition for the rational function

$$\frac{5x+1}{(x+3)(x^2+4x+3)(x^2+4)}$$

(a) $\dfrac{A}{x+3} + \dfrac{B}{(x+3)^2} + \dfrac{C}{x+1} + \dfrac{Dx+E}{x^2+4}$

(b) $\dfrac{A}{x+3} + \dfrac{Bx+C}{x^2+4x+3} + \dfrac{Dx+E}{x^2+4}$

(c) $\dfrac{A}{x+3} + \dfrac{Bx+C}{(x+3)^2} + \dfrac{D}{x+1} + \dfrac{Ex+F}{x^2+4}$

(d) $\dfrac{A}{x+3} + \dfrac{B}{(x+3)^2} + \dfrac{C}{x+1} + \dfrac{D}{x+2} + \dfrac{D}{x-2}$

(e) None of these.

Integration by Partial Fractions (Exercise)

Compute $\displaystyle\int \frac{2x^2+5x-5}{(x+1)(x+3)^2}\,dx$

Evaluate $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)}\,dx$

Find $\displaystyle\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,dx$ showing all necessary work.

Improper integrals

Definition: improper integrals

- integrals with infinite intervals of integration
- integrals with intervals with vertical asymptote (***infinite discontinuities***)

Examples of improper integrals

- Improper integrals with **infinite intervals of integration (Type 1)**:
  - $\int_1^\infty \frac{dx}{x^2}, \int_{-\infty}^0 e^x dx, \int_{-\infty}^\infty \frac{dx}{1+x^2}$
- Improper integrals with **infinite discontinuities** in the interval **(Type 2)**:
  - $\int_{-3}^3 \frac{dx}{x^2}, \int_1^2 \frac{dx}{x-1}, \int_0^\pi \tan x\, dx$
- Improper integrals with **infinite discontinuities** and **infinite intervals of integration (Type 3)**:
  - $\int_0^\infty \frac{dx}{\sqrt{x}}, \int_{-\infty}^\infty \frac{dx}{x^2-9}, \int_1^\infty \sec x\, dx$

- Limit = tool to handle infinity

Improper integrals

Definition: improper integral of f over the interval $[a, \infty]$

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

Definition: improper integral of f over the interval $[-\infty, b]$

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

Definition: improper integral of f over the interval $[-\infty, \infty]$

$$\int_{-\infty}^\infty f(x)\,dx = \int_a^\infty f(x)\,dx + \int_{-\infty}^a f(x)\,dx$$

- In the case where the limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral.
- In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

Improper integrals

Theorem: improper integral of $1/x^p$ over the interval $[1,\infty)$

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & if \;\; p>1 \\ diverges & if \;\; p \le 1 \end{cases}$$

Which of the following statements is true regarding the improper integral $\int_1^\infty \frac{\sin x + 2}{x(x+1)}\,dx$?

(a) The integral converges because $\int_1^\infty \frac{\sin x + 2}{x(x+1)}\,dx \le \int_1^\infty \frac{3}{x^2}\,dx$, which converges.

(b) The integral converges because $\int_1^\infty \frac{\sin x + 2}{x(x+1)}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx$, which converges.

(c) The integral diverges because $\int_1^\infty \frac{\sin x + 2}{x(x+1)}\,dx \ge \int_1^\infty \frac{3}{x}\,dx$, which diverges.

(d) The integral diverges because $\int_1^\infty \frac{\sin x + 2}{x(x+1)}\,dx \ge \int_1^\infty \frac{1}{x}\,dx$, which diverges.

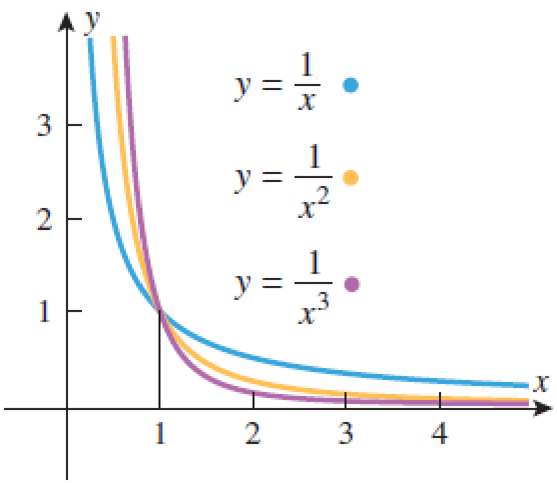
(e) The integral diverges by oscillation.

On $(0,\infty)$

$(\sin x + 2) \le 3$

$x^2 + x > x^2 \Rightarrow \frac{1}{x^2+x} < \frac{1}{x^2}$

$\frac{(\sin x+2)}{x^2+x} \le \frac{3}{x^2}$ where $\int \frac{3}{x^2}\,dx \le \infty$



Improper integrals

Theorem: improper integral of $1/x^p$ over the interval $(0, 1]$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p} & if \;\; p < 1 \\ diverges & if \;\; p \geq 1 \end{cases}$$

Which statement is true about the integral $\int_0^4 \frac{2}{(x-3)^2}\,dx$?

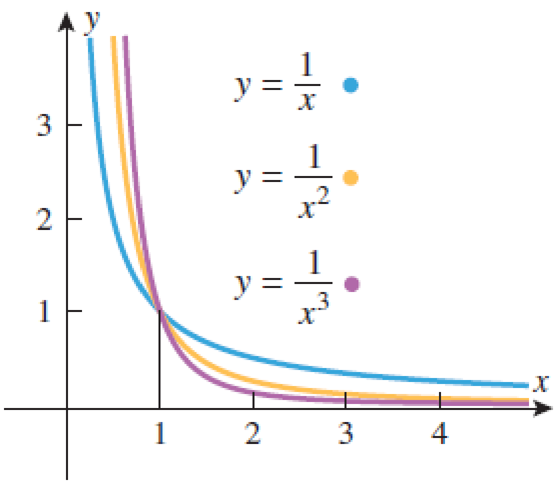
(a) Diverges

(b) Converges to $\frac{8}{3}$

(c) Converges to $\frac{4}{3}$

(d) Converges to $-\frac{8}{3}$

(e) Converges to $-\frac{4}{3}$



Improper integrals

Which of the following statements is true regarding the improper integral $\int_1^\infty \frac{\sin x + 2}{x(x+1)}\,dx$?

(a) The integral converges because $\int_1^\infty \frac{\sin x + 2}{x(x+1)}\,dx \leq \int_1^\infty \frac{3}{x^2}\,dx$, which converges.

(b) The integral converges because $\int_1^\infty \frac{\sin x + 2}{x(x+1)}\,dx \leq \int_1^\infty \frac{1}{x^2}\,dx$, which converges.

(c) The integral diverges because $\int_1^\infty \frac{\sin x + 2}{x(x+1)}\,dx \geq \int_1^\infty \frac{3}{x}\,dx$, which diverges.

(d) The integral diverges because $\int_1^\infty \frac{\sin x + 2}{x(x+1)}\,dx \geq \int_1^\infty \frac{1}{x}\,dx$, which diverges.

(e) The integral diverges by oscillation.

Improper integrals

$$\int_1^\infty xe^{-x^2}\,dx =$$

(a) 1

(b) $2e$

(c) $\frac{1}{2e}$

(d) $\frac{1}{2}$

(e) ∞

Improper integrals

$$\int_0^1 \frac{2}{x^2-1}\,dx =$$

(a) $-\infty$

(b) ∞

(c) $\ln 2$

(d) $-\ln 2$

(e) 0

Improper integrals (Exercise)

Compute $\displaystyle\int_{-1}^{\infty} \frac{1}{1+x^2}\,dx.$

(a) ∞

(b) $\dfrac{\pi}{2}$

(c) $\dfrac{\pi}{4}$

(d) None of these.

(e) $\dfrac{3\pi}{4}$

The improper integral $\displaystyle\int_{1}^{e} \frac{1}{x\ln x}\,dx$

(a) diverges to $-\infty$.

(b) converges to 1.

(c) diverges to ∞.

(d) converges to -1.

(e) converges to $\dfrac{1}{e} - 1$.

Improper integrals (Exercise)

Which of the following statements is true regarding the improper integral $\int_1^\infty \frac{1}{e^x + \sqrt{x}}\, dx$?

(a) The integral converges because $\int_1^\infty \frac{1}{e^x + \sqrt{x}}\, dx < \int_1^\infty \frac{1}{\sqrt{x}}\, dx$ and $\int_1^\infty \frac{1}{\sqrt{x}}\, dx$ converges.

(b) The integral diverges because $\int_1^\infty \frac{1}{e^x + \sqrt{x}}\, dx > \int_1^\infty \frac{1}{e^x}\, dx$ and $\int_1^\infty \frac{1}{e^x}\, dx$ diverges.

(c) The integral diverges because $\int_1^\infty \frac{1}{e^x + \sqrt{x}}\, dx > \int_1^\infty \frac{1}{\sqrt{x}}\, dx$ and $\int_1^\infty \frac{1}{\sqrt{x}}\, dx$ diverges.

(d) The integral converges because $\int_1^\infty \frac{1}{e^x + \sqrt{x}}\, dx < \int_1^\infty \frac{1}{e^x}\, dx$ and $\int_1^\infty \frac{1}{e^x}\, dx$ converges.

(e) The integral converges to 0.

Evaluate $\int_1^\infty \frac{e^{2/x}}{x^2}\, dx$.

(a) $-\frac{1}{2}\left(1 - e^2\right)$

(b) $2\left(1 - e^2\right)$

(c) $-2\left(1 - e^2\right)$

(d) $\frac{1}{2}\left(1 - e^2\right)$

(e) $\frac{1}{2}e^2$

Improper integrals (Exercise)

The integral $\int_0^{\infty} e^{-2x}\,dx$

(a) diverges
(b) converges to 0
(c) converges to $\frac{1}{4}$
(d) converges to $\frac{1}{2}$
(e) converges to 2

Which of the following integrals are improper?

(I) $\int_0^1 \frac{1}{3x-1}\,dx$ (II) $\int_1^3 \ln(x-1)\,dx$ (III) $\int_{-\infty}^1 \frac{1}{x^4}\,dx$

(a) (III) only
(b) (I) and (III) only
(c) (II) and (III) only
(d) (I) and (II) only
(e) All of them are improper.

Improper integrals (Exercise)

Which statement is true about the integral $\int_1^\infty \frac{3\sin^2 x}{x^2}\,dx$?

(a) The integral converges by comparison to $\int_1^\infty \frac{1}{x}\,dx$

(b) The integral diverges by comparison to $\int_1^\infty \frac{3}{x^2}\,dx$

(c) The integral converges by comparison to $\int_1^\infty \frac{3}{x^2}\,dx$

(d) The integral diverges by comparison to $\int_1^\infty \frac{1}{x}\,dx$

(e) None of these

Which of the following statements is true regarding the improper integral $\int_1^\infty \frac{\cos^2 x + 5}{x^4}\,dx$?

(a) It converges since $\int_1^\infty \frac{\cos^2 x + 5}{x^4}\,dx \leq \int_1^\infty \frac{5}{x^4}\,dx$, which converges.

(b) It converges since $\int_1^\infty \frac{\cos^2 x + 5}{x^4}\,dx \leq \int_1^\infty \frac{1}{x^4}\,dx$, which converges.

(c) It converges since $\int_1^\infty \frac{\cos^2 x + 5}{x^4}\,dx \leq \int_1^\infty \frac{6}{x^4}\,dx$, which converges.

(d) It converges to zero.

(e) It diverges by oscillation.