Week 15

Monday, December 2, 2024 7:26 PM

Math 151
Week-In-Review 15

Final Exam Review
Todd Schrader

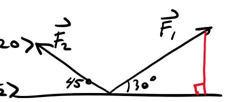
Problem Statements

1. Two forces act on an object. One force has magnitude 40 N in the direction of 30° above the postive x−axis. The other has magnitude 10 in the direction of 45° above the negative x−axis.

(a) Find the magnitude and direction of the resulting force.

$\vec{F_1} = \langle 40\cos(30), 40\sin(30)\rangle = \langle 40\left(\frac{\sqrt{3}}{2}\right), 40\left(\frac{1}{2}\right)\rangle = \langle 20\sqrt{3}, 20\rangle$

$\vec{F_2} = \langle -10\cos(45), 10\sin(45)\rangle = \langle -10\left(\frac{\sqrt{2}}{2}\right), 10\left(\frac{\sqrt{2}}{2}\right)\rangle = \langle -5\sqrt{2}, 5\sqrt{2}\rangle$



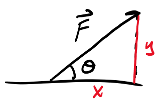
$\vec{F} = \vec{F_1} + \vec{F_2} = \langle 20\sqrt{3} - 5\sqrt{2}, 20 + 5\sqrt{2}\rangle$

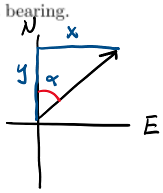
$\boxed{|\vec{F}| = \sqrt{(20\sqrt{3} - 5\sqrt{2})^2 + (20 + 5\sqrt{2})^2}}$

(b) Find the same resulting direction as a bearing.

$\tan(\theta) = \frac{y}{x}$

$\boxed{\theta = \arctan\left(\frac{20 + 5\sqrt{2}}{20\sqrt{3} - 5\sqrt{2}}\right)}$





$\tan(\alpha) = \frac{x}{y}$

$\boxed{\alpha = \arctan\left(\frac{20\sqrt{3} - 5\sqrt{2}}{20 + 5\sqrt{2}}\right) \quad N\,\alpha^\circ E}$

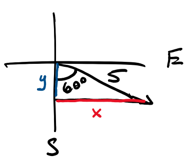
(c) If the object moves from the origin to the point (4, 10) find the work done by the resulting force.

$W = \vec{F} \cdot \vec{D} = |\vec{F}|\,|\vec{D}|\cos(\theta)$

$\vec{D} = (4, 10) - (0, 0) = \langle 4, 10\rangle$

$\boxed{W = (20\sqrt{3} - 5\sqrt{2})\,4 + (20 + 5\sqrt{2})\,10}$

(d) If a wind blows 5 mph at S60°E, find the components of the wind as a vector.



$x = 5\sin(60) = 5\left(\frac{\sqrt{3}}{2}\right)$

$y = -5\cos(60) = -5\left(\frac{1}{2}\right)$

$\boxed{\vec{w} = \left\langle \frac{5\sqrt{3}}{2}, -\frac{5}{2}\right\rangle}$

2. $\boldsymbol{a} = \langle 2, -5\rangle$, $\boldsymbol{b} = \langle 3, 7\rangle$, $\boldsymbol{c} = \langle 10, -2\rangle$.

(a) Find $2\boldsymbol{a} - 3\boldsymbol{b}$.

$$2\langle 2,-5\rangle - 3\langle 3,7\rangle$$

$$\langle 4,-10\rangle - \langle 9, 21\rangle$$

$$= \langle 4-9, -10-21\rangle$$

$$\boxed{2\vec{a} - 3\vec{b} = \langle -5, -31\rangle}$$

(b) Find the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$|\vec{a}| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$$

$$|\vec{b}| = \sqrt{3^2 + 7^2} = \sqrt{58}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos(\theta)$$

$$-29 = \sqrt{29}\sqrt{58} \cos\theta$$

$$\vec{a} \cdot \vec{b} = 2(3) + (-5)(7) = 6 - 35 = -29$$

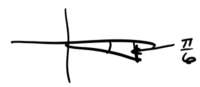
$$\frac{-29}{\sqrt{29}\sqrt{58}} = \cos\theta$$

$$\boxed{\theta = \arccos\left(\frac{-29}{\sqrt{29}\sqrt{58}}\right)}$$

(c) Find the magnitude of the vector projection of $\boldsymbol{c}$ onto $\boldsymbol{b}$.

$$\operatorname{comp}_{\vec{b}} \vec{c} = \frac{\vec{b} \cdot \vec{c}}{|\vec{b}|} = \frac{10(3) + (-2)(7)}{\sqrt{58}} = \boxed{\frac{16}{\sqrt{58}}}$$

$$\operatorname{proj}_{\vec{b}} \vec{c} = \frac{\vec{b} \cdot \vec{c}}{|\vec{b}|} \left(\frac{\vec{b}}{|\vec{b}|}\right) = \frac{16}{\sqrt{58}}\left(\frac{\langle 3,7\rangle}{\sqrt{58}}\right)$$

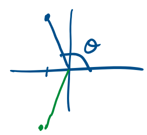


3. Simplify the following:

(a) $\arccos\left(\sin\left(\frac{11\pi}{6}\right)\right) = \underbrace{\arccos\left(\frac{-1}{2}\right)}$

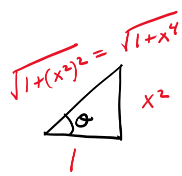
$$\cos(\theta) = \frac{-1}{2}$$

$$\boxed{\theta = \frac{2\pi}{3}}$$

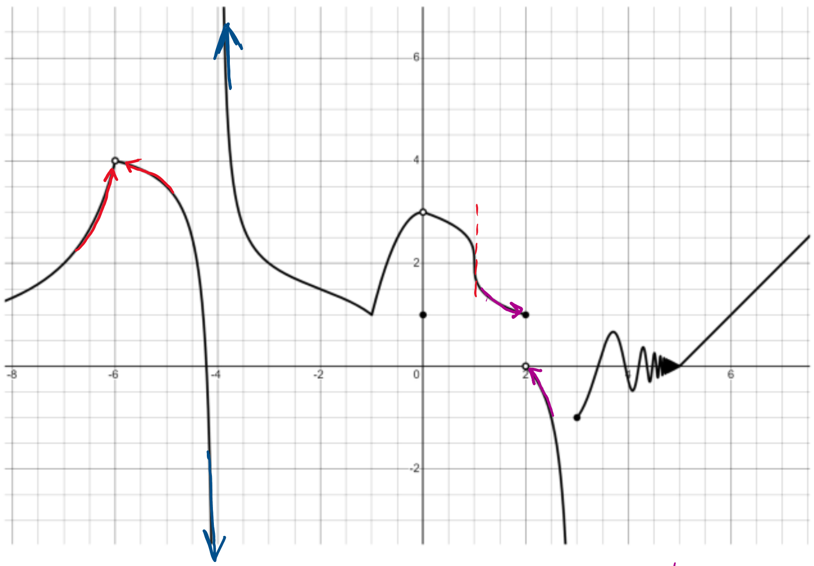


(b) $\sin(\underbrace{\arctan(x^2)}_{\theta}) = \boxed{\frac{x^2}{\sqrt{1+x^4}}}$

$$\tan(\theta) = \frac{x^2}{1}$$



4. Evaluate the limits for the graph of $f(x)$ below.



(a) $\lim_{x \to -6^-} f(x) =$ 4

(b) $\lim_{x \to -6^+} f(x) =$ 4

(c) $\lim_{x \to -6} f(x) =$ 4

(d) $\lim_{x \to -4^-} f(x) =$ $-\infty$

(e) $\lim_{x \to -4^+} f(x) =$ ∞

(f) $\lim_{x \to -4} f(x) =$ DNE

(g) $\lim_{x \to 2^-} f(x) =$ 1

(h) $\lim_{x \to 2^+} f(x) =$ 0

(i) $\lim_{x \to 2} f(x) =$ DNE

5. For the graph above (ignoring $x = 5$),

(a) Where is $f(x)$ not ~~continous~~ continuous?

$x = -6, \quad x = -4, \quad x = 0, \quad x = 2, \quad x = 3$

(b) Where is $f(x)$ not differentiable?

$x = -6, \quad \cancel{x} = -4, \quad x = 0, \quad x = 2, \quad x = 3$

$x = -1 \qquad x = 1$

(I) Not continuous

(II) Corner/Cusp

(III) Vertical Tangent Line

6. Evaluate the limits.

(a) $\lim_{t\to 1} \frac{3t^2+2t-5}{2t^2}$ $= \frac{3+2-5}{2(1)} = \frac{0}{2} = \boxed{0}$

$\frac{\frac{1}{3}-\frac{1}{3}}{3-3} = \frac{0}{0}$ ✓

$\overset{L'H}{=} \lim_{x\to 3} \frac{-\frac{1}{x^2}}{1} = \frac{-1}{3^2} = \boxed{\frac{-1}{9}}$

(b) $\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3}$ $= \lim_{x\to 3} \left(\frac{3-x}{3x}\right)\cdot\frac{1}{x-3} = \lim_{x\to 3} -\frac{\cancel{(x-3)}}{3x}\cdot\frac{1}{\cancel{(x-3)}}$

$= \lim_{x\to 3} \frac{-1}{3x} = \frac{-1}{3(3)} = \boxed{\frac{-1}{9}}$

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$\frac{0}{0}$ ✓

$f(x+h) \quad - \quad f(x) = x^3-5x^2$

(c) $\lim_{h\to 0} \frac{(x+h)^3-5(x+h)^2-x^3+5x^2}{h}$

Def. of Derivative.

$f(x) = x^3-5x^2$

$f'(x) = \boxed{3x^2-10x}$

$\frac{1+1-2}{0} = \frac{0}{0}$ ✓

$\frac{0}{0}$ ✓

(d) $\lim_{x\to 0} \frac{\sin(x)+e^x-2x-1}{x^2}$ $\overset{L'H}{=} \lim_{x\to 0} \frac{\cos(x)+e^x-2}{2x}$

$\overset{L'H}{=} \lim_{x\to 0} \frac{-\sin(x)+e^x}{2} = \frac{-\sin(0)+e^0}{2} = \boxed{\frac{1}{2}}$

7. Consider the function $f(x) = 2\sqrt{x+5} - 4$ $= 2(x+5)^{1/2} - 4$

(a) Find the average rate of change of the function on the interval $[4, 11]$.

$$m = \frac{f(11) - f(4)}{11 - 4}$$

$$f(11) = 2\sqrt{11+5} - 4 = 4$$

$$f(4) = 2\sqrt{4+5} - 4 = 2$$

$$= \frac{4-2}{11-4} = \boxed{\frac{2}{7}}$$

(b) Find c such that $f'(c)$ is equal to your answer from part (a).

M.V.T.

Mean Value Theorem

$$f'(x) = 2 \cdot \frac{1}{2} \cdot (x+5)^{-1/2} = \frac{1}{\sqrt{x+5}} = \frac{2}{7}$$

$$2\sqrt{x+5} = 7$$

$$\sqrt{x+5} = \frac{7}{2}$$

$$x+5 = (3.5)^2$$

$$x = (3.5)^2 - 5 = \boxed{7.25}$$

(c) Show there is a solution to $f(x) = 3$ on the interval $[4, 11]$.

$f(4) = 2$

$f(11) = 4$

$f(x)$ is continuous on $[4, 11]$

therefore true,

I.V.T.

Intermediate Value Theorem.

(d) Set up a limit representing the instantaneous rate of change of $f(x)$ when $x = 4$.

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{\left(2\sqrt{(4+h)+5} - 4\right) - \left(2\sqrt{4+5} - 4\right)}{h}$$

$$f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4}$$

(e) Find the instantaneous rate of change of $f(x)$ when $x = 4$.

$$f'(4) = \frac{1}{\sqrt{4+5}} = \boxed{\frac{1}{3}}$$

Domain: $(-\infty, 0) \cup (0, \infty)$

8. Consider the piecewise function $f(x) = \begin{cases} 5 - \dfrac{1}{x} & \text{if } x \le 1 \\ \dfrac{x^2-1}{x-1} \to x+1 & \text{if } x > 1 \end{cases}$

$5 - \dfrac{1}{x}$, $x \neq 0$ | 1 | $\dfrac{x^2-1}{x-1}$, $\cancel{x \neq 1}$ Not in interval

(a) Where is $f(x)$ not continuous?

$\lim_{x \to 1^-} 5 - \frac{1}{x} = 5 - \frac{1}{1} = 4$

$f(1) = 5 - \frac{1}{1} = 4$

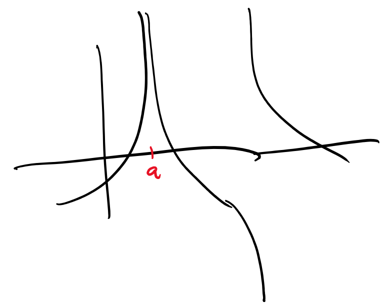
$\lim_{x \to 1^+} \frac{x^2-1}{x-1} = \lim_{x \to 1^+} \frac{(x+1)(x-1)}{(x-1)} = \lim_{x \to 1^+} x+1 = 2$

$\boxed{\text{Not continuous @ } x=0,\ x=1}$

(b) Where is $f(x)$ not differentiable?

$\boxed{x=0,\ x=1}$

$f'(x) = \begin{cases} \frac{1}{x^2} & \text{if } x < 1 \\ 1 & \text{if } x > 1 \end{cases}$

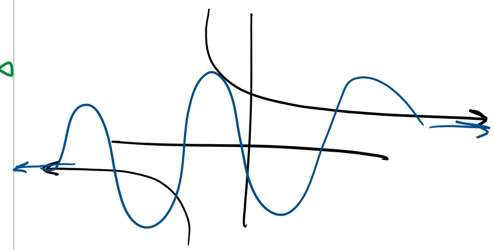


(c) Find any vertical asymptotes of $f(x)$.

V.A. $\boxed{x = 0}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} 5 - \frac{1}{x} = 5 - \frac{1}{0} \to \pm\infty$

$\lim_{x \to 0^+} f(x) = 5 - \frac{1}{0^+} = -\infty$ $\qquad$ $\lim_{x \to 0^-} f(x) = 5 - \frac{1}{0^-} = \infty$



(d) Find any horizontal asymptotes of $f(x)$.

$\lim_{x \to \infty} \frac{x^2-1}{x-1} = \infty$ No H.A.

$\lim_{x \to -\infty} 5 - \frac{1}{x} = 5 - \frac{1}{-\infty} = 5 - 0 = 5$

H.A. $\boxed{y = 5}$

9. Find $\frac{dy}{dx}$ using implicit differentiation.

$(y)^5$

$$\cos(xy) + \log_8(x) = e^{x+y} - y^5$$

$$-\sin(xy)\cdot\left(x\cdot\frac{dy}{dx} + y\cdot 1\right) + \frac{1}{x}\cdot\frac{1}{\ln(8)} = e^{(x+y)}\left(1+\frac{dy}{dx}\right) - 5y^4\cdot\frac{dy}{dx}$$

$$-x\sin(xy)\cdot\frac{dy}{dx} - e^{(x+y)}\cdot\frac{dy}{dx} + 5y^4\cdot\frac{dy}{dx} = y\sin(xy) - \frac{1}{x\ln 8} + e^{(x+y)}$$

$$\boxed{\frac{dy}{dx} = \frac{y\sin(xy) - \frac{1}{x\ln(8)} + e^{x+y}}{-x\sin(xy) - e^{(x+y)} + 5y^4}}$$

10. Find $\frac{d}{dx}\left[x^{\arctan(x)}\right]$

$$y = x^{\arctan(x)}$$

$$\ln(y) = \ln\left(x^{\arctan x}\right) = \arctan(x)\cdot\ln(x)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \arctan(x)\cdot\frac{1}{x} + \ln(x)\cdot\frac{1}{1+x^2}$$

$$\boxed{\frac{dy}{dx} = \left[\frac{\arctan(x)}{x} + \frac{\ln(x)}{1+x^2}\right] x^{\arctan(x)}}$$

11. A bacteria grows at a rate proportional to its size.

$y = y_0 e^{kt}$

(a) If there are 200 cells when $t = 4$ and 1000 cells when $t = 10$, find an equation representing the number of cells at time t.

$(4, 200)$ $(10, 1000)$

$200 = y_0 e^{k\cdot(4)}$

$y_0 = \frac{200}{e^{4k}}$

$\frac{200}{e^{4k}} = \frac{1000}{e^{10k}}$

$200 e^{10k} = 1000 e^{4k}$

$\frac{e^{10k}}{e^{4k}} = \frac{1000}{200}$

$1000 = y_0 e^{k\cdot(10)}$

$y_0 = \frac{1000}{e^{10k}}$

$e^{6k} = 5$

$6k = \ln(5)$

$k = \frac{\ln(5)}{6}$

$1000 = y_0 e^{\frac{\ln(5)}{6}\cdot(10)}$

$y_0 = \frac{1000}{e^{\frac{\ln(5)}{6}\cdot 10}}$

$y_0 = \frac{1000}{5^{5/3}}$

$$\boxed{y(t) = \frac{1000}{5^{5/3}} \cdot e^{\frac{\ln(5)}{6}\cdot t}}$$

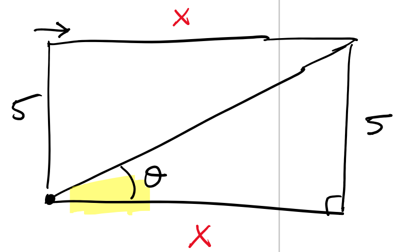
$e^{\frac{\ln(5)}{6}\cdot 10} = e^{\ln(5)\cdot\frac{5}{3}} = e^{\ln\left(5^{5/3}\right)} = 5^{5/3}$

(b) When will there be 20000 cells?

$5x^{-1} \to -5x^{-2}$

$\frac{dx}{dt} = ?$

12. A plane flies at an altitude of 5 km and passes over a a tracking telescope on the ground. When the angle of elevation of the telescope is $\frac{\pi}{3}$, the angle of elevation is decreasing at a rate of $\frac{\pi}{6}$. How fast is the plane traveling at that time?



$$\tan(\theta) = \frac{5}{x} \qquad \frac{d\theta}{dt} = -\frac{\pi}{6}$$

$$\sec^2(\theta) \cdot \frac{d\theta}{dt} = -5x^{-2} \cdot \frac{dx}{dt}$$

$$4\left(\frac{-\pi}{6}\right) = \frac{-5}{(5/\sqrt{3})^2} \cdot \frac{dx}{dt}$$

$$\frac{2\pi}{3} = \frac{3}{5} \quad \frac{dx}{dt}$$

$$\boxed{\frac{10\pi}{9} = \frac{dx}{dt}}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{5}{x}$$

$$\frac{\sqrt{3}/2}{1/2} = \frac{5}{x}$$

$$\sqrt{3} = \frac{5}{x}$$

$$\sqrt{3}x = 5$$

$$x = \frac{5}{\sqrt{3}}$$

$$\frac{1}{(1/2)^2} = 4 = \sec^2\theta$$

13. (a) Use a linear approximation to estimate $\sqrt{630}$.

$f(x) = \sqrt{x} = x^{1/2}$

$a = 625$

Point: $f(625) = \sqrt{625} = 25$

$(625, 25)$

Slope: $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$f'(625) = \frac{1}{2\sqrt{625}} = \frac{1}{50}$

$y - 25 = \frac{1}{50}(x - 625)$

$L(x) = 25 + \frac{1}{50}(x-625)$

$\sqrt{630} = f(630) \approx L(630) = 25 + \frac{1}{50}(630 - 625)$

$= \boxed{25 + \frac{1}{50}(5)}$

(b) The radius of a sphere was measured to be 20 cm with a possible error in measurement of 0.5 cm. What is the maximum possible error in using this value of the radius to calculate the volume of the sphere?

$r = 20 \quad dr = 0.5$

$V = \frac{4}{3}\pi r^3$

$dV = 4\pi r^2 \, dr$

$\boxed{dV = 4\pi (20)^2 (0.5)}$

14. (a) Find the intervals where $f(x) = e^{x^2+6x}$ is increasing/decreasing.

$f'(x) = e^{x^2+6x} \cdot (2x+6) = 0$

$f'(x)$ DNE : N/A

$f'(x) = 0$: $e^{x^2+6x} = 0$ N/A; $2x+6=0$, $2x = -6$, $x = -3$

(-10) (0)

$f'(x)$: $-$ $\quad -3 \quad$ $+$

$f(x)$: $\searrow$ $\nearrow$

$$\boxed{\begin{array}{l} f(x) \text{ Inc}: (-3, \infty) \\ f(x) \text{ Dec}: (-\infty, -3) \end{array}}$$

(b) Determine the locations of any extreme values of $f(x)$.

Local Minimum: @ $x = -3$

No Local Max.

Typo *

(c) Find the intervals where $g(x) = x^4 - x^3 + \overset{-3}{\cancel{}}x^2 - 15x + 10$ is concave up/concave down.

$g'(x) = 4x^3 - 3x^2 + \overset{-6}{\cancel{}}x - 15$

$g''(x) = 12x^2 - 6x + \overset{-6}{\cancel{}} = 0$

$g''(x)$ DNE : N/A

$g''(x) = 0$ $\quad 6(2x^2 - x - \overset{1}{\cancel{}}) = 0$

$6(2x+1)(x-1)$

$x = 1$

$x = -\frac{1}{2}$

(-10) (0) (10)

$f''(x)$: $+$ $\quad -\frac{1}{2} \quad$ $-$ $\quad 1 \quad$ $+$

$f(x)$: $\cup$ $\cap$ $\cup$

$f''(x) = 6(2x+1)(x-1)$

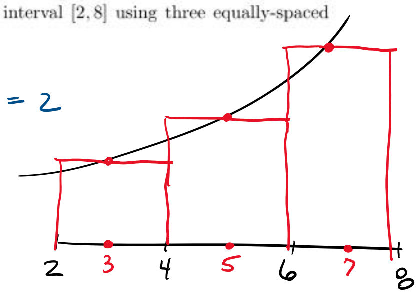
(d) Determine the locations of any inflection points of $g(x)$.

$$\boxed{x = -\frac{1}{2} \text{ and } x = 1}$$

15. Consider the function $f(x) = x^2 + 5$.

(a) Estimate the area under the function on the interval $[2, 8]$ using three equally-spaced rectangles and midpoints.

$$\Delta x = \frac{b-a}{n} = \frac{8-2}{3} = \frac{6}{3} = 2$$



$$2f(3) + 2f(5) + 2f(7)$$

$$2\left[14 + 30 + 54\right]$$

$$2(98) = \boxed{196}$$

* Typo

(b) Set up an ~~integral~~ limit representing the exact area under the curve on the interval $[2, 8]$.

$$\Delta x = \frac{8-2}{n} = \frac{6}{n} \qquad x_i^* = a + i\Delta x = 2 + i\left(\frac{6}{n}\right)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} f\left(2 + \frac{6i}{n}\right)$$

$$= \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} \cdot \left[\left(2 + \frac{6i}{n}\right)^2 + 5\right]}$$

$$\int_2^8 (x^2 + 5)\, dx$$

16. Evaluate

(a) $\int \left(\frac{1}{\sqrt{1-x^2}} + \sin(x) - 5x \right) dx$

$$\boxed{\arcsin(x) + (-\cos x) - 5 \cdot \frac{1}{2} x^2 + C}$$

(b) $\int_{x=1}^{x=3} \frac{2^x}{2^x+3}\, dx$

$u = 2^x + 3$

$du = 2^x \cdot \ln(2)\, dx$

Bounds:

$u(3) = 2^3 + 3 = 11$

$u(1) = 2^1 + 3 = 5$

$$\frac{1}{\ln(2)} \int_1^3 \frac{2^x \ln(2)}{2^x + 3}\, dx$$

$$\frac{1}{\ln(2)} \int_5^{11} \frac{1}{u}\, du = \frac{1}{\ln(2)} \ln|u| \Big|_5^{11} = \boxed{\frac{1}{\ln(2)} \cdot (\ln(11) - \ln(5))}$$

(c) $\frac{d}{dx}\left[\int_{\ln(x)}^{e^{4x}} t \cdot \sec(t)\, dt \right]$

$$\boxed{e^{4x} \cdot \sec(e^{4x}) \cdot e^{4x} \cdot 4 - \ln(x) \cdot \sec(\ln x) \cdot \frac{1}{x}}$$