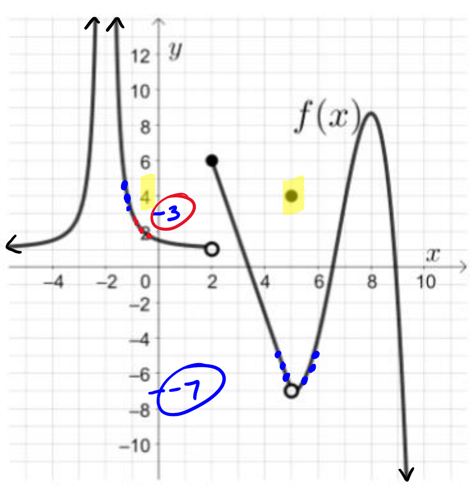
WIR#1

MATHEMATICS TEXAS A&M UNIVERSITY

Section 1.1

- **Left-Hand Limit:** $\lim_{x\to c^-} f(x) = L$ if as x approaches c from the left, $f(x)$ approaches L.
- **Right-Hand Limit:** $\lim_{x\to c^+} f(x) = M$ if as x approaches c from the right, $f(x)$ approaches M.
- **(Two-Sided) Limit:** $\lim_{x\to c} f(x) = L$ if the functional value $f(x)$ approaches L as x approaches c (from either side of c). For a (two-sided) limit to exist, the limit from the left and the limit from the right must both exist and be equal. Otherwise, we say that the (two-sided) limit does not exist.
- Keep in mind that when we are dealing with limits we are only interested in what is going on with the function NEAR $x = c$. What occurs at $x = c$ does not affect the value of the limit.
- The line $x = c$ is a **vertical asymptote** of $f(x)$ if $\lim_{x\to c^-} f(x) \to \pm\infty$ or $\lim_{x\to c^+} f(x) \to \pm\infty$. These limits are referred to as **infinite limits.**
- In this section, we learned how to estimate a limit numerically and from a graph.
- Keep in mind that when stating our answer involving limits, we use an equal sign only if the limit exists (approaches a finite number). If the function increases or decreases without bound as we approach the particular value of x we state the limit "Does Not Exist" but we can sometimes describe the way in which it does not exist by using an arrow and $\pm\infty$.

1. Use the graph below to find each of the following, if it exists.



(a) $\lim_{x\to 2^-} f(x) = \boxed{1}$

(b) $\lim_{x\to 2^+} f(x) = \boxed{6}$

(c) $\lim_{x\to 2} f(x)$ $\boxed{\text{DNE}}$

(d) $\lim_{x\to -1} f(x) = \boxed{3}$

(e) $\lim_{x\to 5} f(x) = \boxed{-7}$

(f) $f(5) = \boxed{4}$

(g) $\lim_{x\to -2^-} f(x)$ DNE

$\lim_{x\to -2^-} f(x) \to \infty$

(h) $\lim_{x\to -2^+} f(x)$ DNE

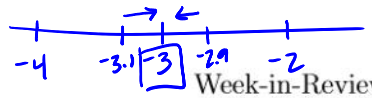
$\lim_{x\to -2^+} f(x) \to \infty$

(i) $\lim_{x\to -2} f(x)$ DNE

$\lim_{x\to -2} f(x) \to \infty$

(j) For what value(s) of k does $\lim_{x\to k} f(x)$ not exist?

$k=2,\ k=-2$



Y_1

2. Let $f(x) = \dfrac{4x^3 - 3x^5 - 3}{2x^2 - x - 21}$. Complete the following table and use it to estimate the limits below numerically. ~~If the limit~~ does not exist, describe the way in which ~~is~~ it does not exist. If needed, round your values to four decimal places.

x	$f(x)$	x	$f(x)$
-3.1	558.1140	-2.9	-402.1707
-3.01	4832.1633	-2.99	-4676.4802
-3.001	47616.3494	-2.999	-47460.6689

(a) $\lim_{x\to -3^-} f(x)$ DNE

$\lim_{x\to -3^-} f(x) \to \infty$

(b) $\lim_{x\to -3^+} f(x)$ DNE

$\lim_{x\to -3^+} f(x) \to -\infty$

(c) $\lim_{x\to -3} f(x)$ DNE

(d) What can we conclude is occurring on the graph of $f(x)$ at $x = -3$?

$x = -3$ is a vertical asymptote

cut-off #'s

3. Let $f(x) = \begin{cases} ① \ 10 - x^2, & \text{if } x < 5 \\ ② \ 3x + 7, & \text{if } x \geq 5 \end{cases}$

Use a table of values to estimate $\lim_{x\to 5} f(x)$.

x	$f(x) = 10 - x^2$
4.9	-14.01
4.99	-14.9001
4.999	-14.99001

x	$f(x) = 3x + 7$
5.1	22.3
5.01	22.03
5.001	22.003

$\lim_{x\to 5^-} f(x) = -15$

$\lim_{x\to 5^+} f(x) = 22$

$\lim_{x\to 5} f(x)$ Does Not Exist

Section 1.2

- Let f and g be two functions, and assume that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exists. Then

  1. $\lim_{x\to c} k = k$ for any constant k
  2. $\lim_{x\to c} x = c$
  3. $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
  4. $\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
  5. $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$ for any constant k
  6. $\lim_{x\to c}[f(x)\cdot g(x)] = [\lim_{x\to c} f(x)][\lim_{x\to c} g(x)]$
  7. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$
  8. $\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$ where n is a positive integer
  9. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}, L > 0$ for n even.

- To evaluate $\lim_{x\to c} \frac{f(x)}{g(x)}$ in this class, try direction substitution to evaluate both the limit of the numerator and the limit of the denominator. There are three possibilities:

  $\frac{\text{non-zero \#}}{\text{non-zero \#}}$ 1. If you get a nonzero number as a result for both the limit of the numerator and the limit of the denominator, your answer is just the quotient of the two numbers. Note: If you are dealing with a piece-wise defined function, you may need to investigate a little further.

  $\frac{0}{0}$ 2. If you get zero as a result for both the limit of the numerator and the limit of the denominator, the requested limit is in indeterminate form (i.e. $\frac{0}{0}$). In this case, you must algebraically manipulate (factor, conjugate, common denominator, write absolute value as a piece-wise), cancel, and then do direct substitution again.

  $\frac{\text{nonzero\#}}{0}$ 3. If you get a nonzero number as a result for the limit of the numerator and zero as a result for the limit of the denominator, (i.e. $\frac{nonzero\#}{0}$), then the limit does not exist and you have an infinite limit. You can evaluate the function on either side of $x = c$ to further describe the behavior. You can also conclude that $x = c$ is a vertical asymptote.

5. If $\lim_{x\to 2} f(x) = 10$ and $\lim_{x\to 2} g(x) = -8$, find the following:

   Prop 3 & 4

   Prop 5

   (a) $\lim_{x\to 2}(5g(x) - f(x) + x - 4)$ $= 5\cdot\underbrace{\lim_{x\to 2} g(x)}_{-8} - \underbrace{\lim_{x\to 2} f(x)}_{10} + \underbrace{\lim_{x\to 2} x}_{2\ (\text{Prop 2})} - \underbrace{\lim_{x\to 2} 4}_{4\ (\text{Prop 1})}$

   $= 5(-8) - 10 + 2 - 4$

   $= \boxed{-52}$

   (b) $\lim_{x\to 2} \frac{f(x)+10}{g(x)-x^2}$

   $\lim_{x\to 2}(f(x)+10) = \underbrace{\lim_{x\to 2} f(x)}_{10} + \underbrace{\lim_{x\to 2} 10}_{10} = 20$

   $\lim_{x\to 2}(g(x)-x^2) = \underbrace{\lim_{x\to 2} g(x)}_{-8} - \underbrace{\lim_{x\to 2} x^2}_{(\underbrace{\lim_{x\to 2} x}_{2})^2} = -8-(2)^2 = -8-4 = -12$

   $\Rightarrow \lim_{x\to 2} \frac{f(x)+10}{g(x)-x^2} = \frac{\lim_{x\to 2}(f(x)+10)}{\lim_{x\to 2}(g(x)-x^2)} = \frac{20}{-12} = \boxed{-\frac{5}{3}}$

6. Evaluate the limits algebraically. If the limit does not exist, state so and use limit notation to describe the infinite behavior. For each function, describe what is occurring on the graph of the function at the value of x that the limit is approaching.

(a) $\lim_{x\to 3} \frac{x^2+5x+6}{4x^2+13x+3}$

$\lim_{x\to 3}(x^2+5x+6) = 3^2+5(3)+6 = 30$

$\lim_{x\to 3}(4x^2+13x+3) = 4(3)^2+13(3)+3 = 78$

$= \frac{30}{78}$

$= \boxed{\frac{5}{13}}$

A filled-in circle at $(3, 5/13)$

(b) $\lim_{x\to -9} \frac{x^2-81}{3x^2+20x-63}$

$\lim_{x\to -9}(x^2-81) = (-9)^2-81 = 81-81 = 0$

$\lim_{x\to -9}(3x^2+20x-63) = 3(-9)^2+20(-9)-63 = 0$

$\frac{0}{0}$ Indeterminate Form ⇒ Do Algebra! Here, we factor!

$= \lim_{x\to -9} \frac{\cancel{(x+9)}(x-9)}{\cancel{(x+9)}(3x-7)}$

$= \lim_{x\to -9} \frac{x-9}{3x-7}$

$= \frac{-9-9}{3(-9)-7} = \frac{-18}{-34} = \boxed{\frac{9}{17}}$

A hole at $(-9, 9/17)$

(c) $\lim_{x\to -3} \frac{\sqrt{x^2+16}-5}{x+3}$

$\lim_{x\to -3}(\sqrt{x^2+16}-5) = \sqrt{(-3)^2+16}-5 = 5-5 = 0$

$\lim_{x\to -3}(x+3) = -3+3 = 0$

$\frac{0}{0}$ Indeterminate Form ⇒ Do Algebra! Here, we multiply by the conjugate!

$(\sqrt{x^2+16}-5)(\sqrt{x^2+16}+5)$ FOIL!

$= \underbrace{\sqrt{x^2+16}\sqrt{x^2+16}}_{x^2+16} \cancel{+5\sqrt{x^2+16}} \cancel{-5\sqrt{x^2+16}} - \underbrace{5(5)}_{25}$

$= x^2+16-25 = x^2-9$

$\lim_{x\to -3} \frac{(\sqrt{x^2+16}-5)(\sqrt{x^2+16}+5)}{x+3} \cdot \frac{1}{(\sqrt{x^2+16}+5)}$

$= \lim_{x\to -3} \frac{x^2-9}{(x+3)(\sqrt{x^2+16}+5)}$

$= \lim_{x\to -3} \frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}(\sqrt{x^2+16}+5)}$

$= \lim_{x\to -3} \frac{x-3}{\sqrt{x^2+16}+5} = \frac{-3-3}{\sqrt{(-3)^2+16}+5} = \frac{-6}{5+5} = \frac{-6}{10} = \boxed{-3/5}$

A hole at $(-3, -3/5)$

(d) $\lim_{x\to 3^-} \frac{x^2+3x}{(x-3)^2}$

$f(x)$

$\lim_{x\to 3^-}(x^2+3x) = 3^2+3(3) = 9+9 = 18$

$\lim_{x\to 3^-}(x-3)^2 = (3-3)^2 = 0^2 = 0$

$\frac{18}{0}$ $\frac{\text{Nonzero \#}}{0} \Rightarrow$ $\boxed{\text{Limit does not exist, infinite limit, vertical asymptote at } x=3}$

We know the limit is tending towards $+\infty$ or $-\infty$, to determine which: we plug in a value near 3 (on the left)

$f(2.999) = 1799\,1001$

$\frac{1}{9}-\frac{1}{5}$

$\boxed{\lim_{x\to 3^-}\frac{x^2+3x}{(x-3)^2} \text{ DNE}}$ $\boxed{\lim_{x\to 3^-}\frac{x^2+3x}{(x-3)^2} \to \infty}$

(e) $\lim_{x\to 9} \frac{\frac{1}{9}-\frac{1}{x}}{9-x}$

$\lim_{x\to 9} \frac{1}{9}-\frac{1}{x} = \frac{1}{9}-\frac{1}{9} = 0$

$\lim_{x\to 9} (9-x) = 9-9 = 0$

$\frac{0}{0}$ (Indeterminate Form)

$\Rightarrow$ Do Algebra!

Here, we get a common denominator!

$= \lim_{x\to 9} \frac{\frac{x\cdot 1}{x\cdot 9}-\frac{1\cdot 9}{x\cdot 9}}{9-x}$

$= \lim_{x\to 9} \frac{\frac{x}{9x}-\frac{9}{9x}}{9-x}$

$= \lim_{x\to 9} \frac{\frac{x-9}{9x}}{9-x}$

$= \lim_{x\to 9} \frac{x-9}{9x}\cdot\frac{1}{9-x} = \lim_{x\to 9}\frac{x-9}{9x(9-x)} = \lim_{x\to 9}\frac{-1(-x+9)}{9x(9-x)} = \lim_{x\to 9}\frac{-1\cancel{(9-x)}}{9x\cancel{(9-x)}} = \lim_{x\to 9}\frac{-1}{9x} = \frac{-1}{9(9)} = \boxed{\frac{-1}{81}}$

$\boxed{\text{A hole at } (9, -1/81)}$

7. Algebraically evaluate the limit below. Assume A is a real number such that $A > 0$.

$$\lim_{x\to -2} \frac{Ax^2+2Ax}{x^2+x-2}$$

$\lim_{x\to -2}(Ax^2+2Ax) = A(-2)^2+2A(-2) = 4A-4A = 0$

$\lim_{x\to -2}(x^2+x-2) = (-2)^2+(-2)-2 = 4-4 = 0$

$\frac{0}{0}$ (Indeterminate Form) $\Rightarrow$ Do Algebra!

Here, we factor!

$= \lim_{x\to -2} \frac{Ax\cancel{(x+2)}}{\cancel{(x+2)}(x-1)}$

$= \lim_{x\to -2} \frac{Ax}{(x-1)}$

$= \frac{A(-2)}{-2-1} = \frac{-2A}{-3} = \boxed{\frac{2A}{3}}$

8. Given $f(x)$ below, find each of the following. If the limit does not exist, state so and use limit notation to describe the infinite behavior.

$$f(x) = \begin{cases} ① \dfrac{x^2+x}{3-x}, & \text{if } x < 7 \\ ② \dfrac{-16x}{4x-20}, & \text{if } 7 < x \leq 10 \\ ③ 5^x - 4x^3, & \text{if } x > 10 \end{cases}$$

7 & 10 are cut-off #'s

(a) $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} \dfrac{x^2+x}{3-x}$

Since $x=3$ is on the interval $x<7$ we use Rule 1

$\boxed{\text{DNE}}$

$\lim\limits_{x\to 3^+} (x^2+x) = 3^2+3 = 12$

$\lim\limits_{x\to 3^+} (3-x) = 3-3=0$

$\dfrac{\text{Non-zero \#}}{0} \Rightarrow$ the limit does not exist, infinite limit, $x=3$ is a vertical asymptote

$f(3.0001) = \dfrac{3.0001^2+3.0001}{3-3.0001} = -120007$

$\boxed{\lim\limits_{x\to 3^+} f(x) \to -\infty}$

(b) $\lim\limits_{x\to 7} f(x)$

Since $x=7$ is a cutoff # we must look at the one-sided limits.

$\lim\limits_{x\to 7^-} f(x) = \lim\limits_{x\to 7^-} \dfrac{x^2+x}{3-x} = \dfrac{7^2+7}{3-7} = \dfrac{56}{-4} = -14$

$\lim\limits_{x\to 7^+} f(x) = \lim\limits_{x\to 7^+} \dfrac{-16x}{4x-20} = \dfrac{-16(7)}{4(7)-20} = \dfrac{-112}{8} = -14$

Therefore, $\boxed{\lim\limits_{x\to 7} f(x) = -14}$

(c) $\lim\limits_{x\to 5} f(x) = \lim\limits_{x\to 5} \dfrac{x^2+x}{3-x} = \dfrac{5^2+5}{3-5} = \dfrac{30}{-2} = \boxed{-15}$

Since $x=5$ is on the interval $x<7$, we use rule 1.