If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then the cross product of $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a}\times\mathbf{b}$, is defined as

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

The cross product $\mathbf{a}\times\mathbf{b}$ is a vector orthogonal to both vectors $\mathbf{a}$ and $\mathbf{b}$.

Example 1. *True/False.*
(a) The cross product $\mathbf{v}\times\mathbf{v} = 0$ *for any vector* $\mathbf{v}$.

True *False*

(b) Suppose $\mathbf{a}$ *and* $\mathbf{b}$ *are nonzero vectors. Then* $\mathbf{a}\times\mathbf{b} \neq \mathbf{b}\times\mathbf{a}$. *However,* $|\mathbf{a}\times\mathbf{b}| = |\mathbf{b}\times\mathbf{a}|$.

True *False*

(c) Suppose $\mathbf{a}$ *and* $\mathbf{b}$ *are nonzero vectors. Then* $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2$.

True *False*

(d) Consider a line given by parametric equations $x + 2 = \dfrac{y-2}{3} = \dfrac{1-z}{5}$. *The vector* $\langle 1, 3, 5\rangle$ *is a direction vector of the line.*

True *False*

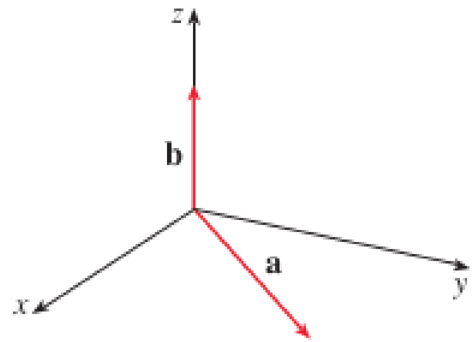
(e) Consider that a plane P_1 *is orthogonal to the line given by equations*

$$x = -1 + t,\ y = 2t,\ z = 3 - 3t.$$

Then $\mathbf{v} = \langle 2, 4, -6\rangle$ *is a normal vector to the plane* P_1 .

True *False*

Example 2 (12.4). *As shown in the figure below, let the vector* $\mathbf{a}$ *lies in the* xy*-plane and* $\mathbf{b}$ *is a vector in the direction of* $\mathbf{k}$. *Suppose* $|\mathbf{a}| = 5$ *and* $|\mathbf{b}| = 4$.



(I) Compute $|\mathbf{a} \times \mathbf{b}|$.

(II) Use the right hand rule to determine whether the components of $\mathbf{a} \times \mathbf{b}$ *are positive, negative, or zero.*

Example 3 (12.4). *Consider a triangle* ABC *with vertices* $A(0, 0, 0)$, $B(1, 2, 0)$, *and* C *on the* y*-axis. If the area of the triangle is* $3/2$, *determine the coordinates of the vertex* C.

Example 4 (12.4). *Find unit vectors orthogonal to the plane passing through the points* $A(1, 0, 0)$, $B(0, 1, 2)$ *and* $C(1, 1, 3)$.

Example 5 (12.4). *Determine whether the points* $A(0, 1, 2)$, $B(2, 1, 0)$, $C(-1, 4, -3)$, *and* $D(4, 1, -2)$ *lie in the same plane.*

If P and Q are two points on a line L, then the **vector equation of the line segment from P to Q is given by**

$$\langle x, y, z\rangle = \overrightarrow{OP} + t(\overrightarrow{PQ}) = (1-t)\overrightarrow{OP} + t\,\overrightarrow{OQ} \qquad 0 \le t \le 1$$

Example 6 (12.5)**.** *Suppose the line L_1 passes through the point $P(2, 2, 5)$ and is parallel to the line*

$$L_2: \;\; x + 5 = \frac{y+1}{3} = \frac{z-2}{2}$$

(a) Determine symmetric equations of the line L_1.

(b) Find the point of intersection of the line L_1 and the yz-plane (if any).

(c) Show that the line L_1 passes through the point $Q(3, 5, 7)$, and determine an equation of the line segment from P to Q.

Example 7 (12.5)**.** *Determine whether the lines L_1 and L_2 are parallel, intersecting, or skew. If they intersect, find the point of intersection.*

$$L_1: \quad \frac{x-1}{3} = \frac{y+3}{8} = z$$

$$L_2: \quad \frac{x-3}{-2} = \frac{y-1}{-4} = \frac{z-4}{-4}.$$

Example 8 (12.5). *Determine an equation of the plane passing through the points* $P(1, 1, 3)$ *and* $Q(2, 1, 5)$ *and is perpendicular to the plane* $y = 2x + 3z - 4$.

The distance D *from the point* $P_1(x_1, y_1, z_1)$ *to the plane* $ax + by + cz + d = 0$ *is*

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 9 (12.5). *Find the distance between the parallel planes* $4x + 2y - z = 1$ *and* $8x + 4y - 2z = 7$.

Example 10 (12.5)**.** *Consider two planes* $3x - y + z = 2$ *and* $2x + y - z = 3$.

(a) Find the acute angle between these two planes.

(b) Find an equation of the line of intersection L *of these two planes.*

Example 11 (12.6). *Identify and sketch the following quadric surfaces.*

(a) $x^2 = y^2 + z^2$

(b) $x^2 + y^2 - 2x - 2y - z + 5 = 0$

(c) $9x^2 - 9y^2 + z^2 - 9 = 0$