Math 151 - Week-In-Review 3

Topics for the week:

2.3 Calculating Limits Using Limit Laws
2.5 Continuity
2.6 Limits at Infinity and Horizontal Asymptotes

2.2 **Cont. Limits**

1. Evaluate the limit, $\lim_{x\to 3^-}\left(\frac{1-x}{2x-6}\right)$, if the limit exists.

$$\lim_{x\to 3^-}\left(\frac{\overbrace{1-x}^{\to -2}}{\underbrace{2x-6}_{\to 0^-}}\right) = \infty$$

3^- means	$1-x$	$2x-6$
2.9	-1.9	-0.2
2.99	-1.99	-0.02
2.999	-1.999	-0.002
	$\to -2$	$\to 0^-$

This tells us $g(x) = \frac{1-x}{2x-6}$ has a vertical asymptote at $x=3$.

2.3 Calculating Limits Using Limit Laws

2. Given $\lim_{x\to 4}(f(x)) = -8$, $\lim_{x\to 4}(g(x)) = 22$, $\lim_{x\to 4}(2h(x)) = 6$, and $\lim_{x\to 4^-}(k(x)) = 0$, determine the value of each limit, if the limit exists.

(a) $\lim_{x\to 4}\left(\frac{\sqrt{g(x)+h(x)}}{3f(x)}\right) = \frac{\lim_{x\to 4}\left(\sqrt{g(x)+h(x)}\right)}{\lim_{x\to 4}(3f(x))}$

$$= \frac{\sqrt{\lim_{x\to 4}(g(x)) + \lim_{x\to 4}(h(x))}}{3\lim_{x\to 4}(f(x))}$$

$$= \frac{\sqrt{22+3}}{3\cdot(-8)} = \boxed{\frac{5}{-24}}$$

Note:
$\lim_{x\to 4}[2h(x)] = 6$
$2\lim_{x\to 4}[h(x)] = 6$
$\lim_{x\to 4}(h(x)) = 3$

(b) $\lim_{x\to 4^-}\left(\frac{1}{2}f(x) - k(x)\right) = \lim_{x\to 4^-}\left(\frac{1}{2}f(x)\right) - \lim_{x\to 4^-}(k(x))$

$$= \frac{1}{2}\lim_{x\to 4^-}(f(x)) - \lim_{x\to 4^-}(k(x))$$

$$= \frac{1}{2}(-8) - (0)$$

$$= -4$$

Note if:
$\lim_{x\to 4} f(x) = -8$
then $\lim_{x\to 4^-} f(x) = -8$

3. Evaluate the limit, $\lim_{x \to 0} \left((x+1)^{1996}\right)$, if it exists.

$$\lim_{x \to 0} \left[(x+1)^{1996}\right] = (0+1)^{1996}$$
$$= 1$$

4. Evaluate the limit, $\lim_{t \to 3} \left(\dfrac{t^2 - 2t - 3}{t^2 - 9}\right)$, if it exists.

$$\lim_{t \to 3} \left(\frac{\overbrace{t^2 - 2t - 3}^{\to 0}}{\underbrace{t^2 - 9}_{\to 0}}\right) = \lim_{t \to 3} \left(\frac{(t-3)(t+1)}{(t-3)(t+3)}\right)$$
$$= \lim_{t \to 3} \left(\frac{t+1}{t+3}\right)$$
$$= \frac{3+1}{3+3}$$
$$= \frac{4}{6}$$
$$= \frac{2}{3}$$

5. Evaluate the limit, $\lim_{x \to -1} \left(\dfrac{x^2 - 1}{x^2 - 5x + 6}\right)$, if it exists.

$$\lim_{x \to -1} \left(\frac{\overbrace{x^2 - 1}^{\to 0}}{\underbrace{x^2 - 5x + 6}_{\to 1+5+6=12}}\right) = 0$$

· Make sure to consider the behavior of the numerator and denominator before doing more work.

6. Evaluate the limit, $\lim_{h\to 0}\left(\dfrac{\sqrt{3h+7}-\sqrt{7}}{h}\right)$, if it exists.

$$\lim_{h\to 0}\left(\frac{\overbrace{\sqrt{3h+7}-\sqrt{7}}^{\to 0}}{\underbrace{h}_{\to 0}}\right) = \lim_{h\to 0}\left[\frac{(\sqrt{3h+7}-\sqrt{7})}{h}\cdot\frac{(\sqrt{3h+7}+\sqrt{7})}{(\sqrt{3h+7}+\sqrt{7})}\right]$$

Multiply by the conjugate of the numerator.

$$= \lim_{h\to 0}\left[\frac{(3h+7)-(\sqrt{3h+7})(\sqrt{7})+(\sqrt{3h+7})(\sqrt{7})-7}{h(\sqrt{3h+7}+\sqrt{7})}\right]$$

$$= \lim_{h\to 0}\left[\frac{3h+7-7}{h(\sqrt{3h+7}+\sqrt{7})}\right]$$

$$= \lim_{h\to 0}\left[\frac{3h}{h(\sqrt{3h+7}+\sqrt{7})}\right]$$

$$= \lim_{h\to 0}\left[\frac{3}{\sqrt{3h+7}+\sqrt{7}}\right]$$

$$= \frac{3}{\sqrt{7}+\sqrt{7}} = \frac{3}{2\sqrt{7}}$$

7. Evaluate the limit, $\lim_{x\to -6}\left(\dfrac{x^{-1}+\frac{1}{6}}{x+6}\right)$, if it exists.

$$\lim_{x\to -6}\left(\frac{x^{-1}+\frac{1}{6}}{x+6}\right) = \lim_{x\to -6}\left(\frac{\overbrace{\frac{1}{x}+\frac{1}{6}}^{\to 0}}{\underbrace{x+6}_{\to 0}}\right)$$

Multiply by the common denominator of the numerator

$$= \lim_{x\to -6}\left[\frac{(\frac{1}{x}+\frac{1}{6})}{(x+6)}\cdot\frac{(6x)}{(6x)}\right]$$

$$= \lim_{x\to -6}\left[\frac{6+x}{6x(x+6)}\right]$$

$$= \lim_{x\to -6}\left[\frac{1}{6x}\right]$$

$$= \frac{1}{-36}$$

8. For $f(x) = \dfrac{|x-9|}{2x^2-17x-9}$, evaluate $\lim\limits_{x\to 9^-} f(x)$ and $\lim\limits_{x\to 9^+} f(x)$. Does $\lim\limits_{x\to 9} f(x)$ exist?

recall $|x-9| = \begin{cases} -(x-9) & \text{if } x<9 \\ x-9 & \text{if } x\geq 9 \end{cases}$

$$\lim_{x\to 9^-} f(x) = \lim_{x\to 9^-}\left[\frac{\overbrace{-(x-9)}^{\to 0}}{\underbrace{2x^2-17x-9}}\right]$$
$$= \lim_{x\to 9^-}\left[\frac{-(x-9)}{(2x+1)(x-9)}\right]^{\to 0}$$
$$= \lim_{x\to 9^-}\left[\frac{-1}{2x+1}\right]$$
$$= -1/19$$

$$\lim_{x\to 9^+} f(x) = \lim_{x\to 9^+}\left[\frac{(x-9)}{(2x+1)(x-9)}\right]$$
$$= \lim_{x\to 9^+}\left[\frac{1}{2x+1}\right]$$
$$= 1/19$$

Since $\lim\limits_{x\to 9^-} f(x) \neq \lim\limits_{x\to 9^+} f(x)$

$\lim\limits_{x\to 9} f(x)$ does not exist

9. For $f(x) = \begin{cases} x^2-36 & x<6 \\ \ln(x-5) & x\geq 6 \end{cases}$, evaluate $\lim\limits_{x\to 6^-} f(x)$ and $\lim\limits_{x\to 6^+} f(x)$. Does $\lim\limits_{x\to 6} f(x)$ exist?

6^- means $x<6$

$$\lim_{x\to 6^-} f(x) = \lim_{x\to 6^-}\left(x^2-36\right)$$
$$= 0$$

6^+ means $x>6$

$$\lim_{x\to 6^+} f(x) = \lim_{x\to 6^+}\left(\ln(x-5)\right)$$
$$= \ln(1)$$
$$= 0$$

Since $\lim\limits_{x\to 6^-} f(x) = \lim\limits_{x\to 6^+} f(x)$, $\lim\limits_{x\to 6} f(x) = 0$

10. Evaluate the limit, $\lim_{x\to-\pi^+}\left((x+\pi)\sin\left(\frac{1}{x+\pi}\right)\right)$, if it exists.

For $\theta \in (-\infty,\infty)$, $-1 \le \sin(\theta) \le 1$, so for $x > -\pi$

$$-1 \le \sin\left(\frac{1}{x+\pi}\right) \le 1.$$

Since $x > -\pi$, $x+\pi > 0$ so

$$-1(x+\pi) \le (x+\pi)\sin\left(\frac{1}{x+\pi}\right) \le (x+\pi)(1)$$

Apply the limit as $x \to -\pi$ to all parts of the inequality.

$$\lim_{x\to-\pi}(-1(x+\pi)) \le \lim_{x\to-\pi}\left[(x+\pi)\sin\left(\frac{1}{x+\pi}\right)\right] \le \lim_{x\to-\pi}(x+\pi)$$

$$0 \le \lim_{x\to-\pi}\left[(x+\pi)\sin\left(\frac{1}{x+\pi}\right)\right] \le 0$$

Thus by the Squeeze Theorem, $\lim_{x\to-\pi}\left[(x+\pi)\sin\left(\frac{1}{x+\pi}\right)\right] = 0.$

2.5 Continuity

11. Determine whether $f(x) = \begin{cases} \frac{2x^2+x-6}{x+2} & \text{if } x < -2 \\ -7 & \text{if } x = -2 \\ \sqrt{3x+6} & \text{if } x > -2 \end{cases}$ is continuous at $x = -2$.

1. $f(-2) = -7$

2. $\lim_{x\to-2^-}(f(x)) = \lim_{x\to-2^-}\left[\frac{2x^2+x-6}{x+2}\right]$ ($\to 0$)

$= \lim_{x\to-2^-}\left[\frac{(2x-3)(x+2)}{x+2}\right]$ ($\to 0$)

$= \lim_{x\to-2^-}(2x-3)$

$= 2(-2)-3$

$= -7$

or $\lim_{x\to-2^+}(f(x)) = \lim_{x\to-2^+}\left[\sqrt{3x+6}\right]$

$= \sqrt{3(-2)+6}$

$= 0$

Since $\lim_{x\to-2^-}(f(x)) \ne \lim_{x\to-2^+}(f(x))$, $\lim_{x\to-2}(f(x))$ does not exist

So $f(x)$ is not continuous at $x=-2$

12. Determine the values where $f(x) = \dfrac{x^2+2x}{x^4-3x^3-10x^2}$ is not continuous. Then classify the value(s) as a vertical asymptote or removable discontinuity.

$f(x) = \dfrac{x(x+2)}{x^2(x^2-3x-10)} = \dfrac{x(x+2)}{x^2(x-5)(x+2)}$

Check: $x = 0, 5, -2$
based on $x^2(x-5)(x+2)=0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left[\dfrac{x(x+2)}{x^2(x-5)(x+2)}\right] = \lim_{x\to 0^-}\left[\dfrac{\overbrace{1}^{\to 1}}{\underbrace{x(x-5)}_{\to 0^+}}\right] = \infty$

so $x=0$ is a vertical asymptote on the graph of $f(x)$

$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+}\left[\dfrac{x(x+2)}{x^2(x-5)(x+2)}\right] = \lim_{x\to -2^+}\left[\dfrac{\overbrace{1}^{\to 1}}{\underbrace{x(x-5)}_{\to 14}}\right] = \dfrac{1}{14}$

so $f(x)$ has a removable discontinuity

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+}\left[\dfrac{x(x+2)}{x^2(x-5)(x+2)}\right] = \lim_{x\to 5^+}\left[\dfrac{\overbrace{1}^{\to 1}}{\underbrace{x(x-5)}_{\to 0^+}}\right] = \infty$

so $x=5$ is a vertical asymptote on the graph of $f(x)$

We do not need the limit on both sides

13. Determine the value(s) of a and b that will make $g(x) = \begin{cases} \frac{x^2+3x-10}{x-2} & \text{if } x < 2 \\ 4ax - 3b & \text{if } 2 \le x < 5 \\ e^{5-x} & \text{if } x \ge 5 \end{cases}$

a continuous function.

1. $f(2) = 4a(2) - 3b = 8a - 3b$

2. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left[\dfrac{(x+5)\overbrace{(x-2)}^{\to 0}}{\underbrace{x-2}_{\to 0}}\right]$

$= \lim_{x\to 2^-}(x+5)$

$= 7$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(4ax-3b)$

$= 8a-3b$

$\lim_{x\to 2} f(x)$ exists if $7 = 8a-3b$

3. $f(x)$ is continuous at $x=2$ if
$7 = 8a-3b$

1. $f(5) = e^{5-5} = e^0 = 1$

2. $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-}(4ax-3b)$

$= 20a - 3b$

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+}(e^{5-x})$

$= e^0$

$= 1$

$\lim_{x\to 5} f(x)$ exists if $1 = 20a-3b$

3. $f(x)$ is continuous at $x=5$ if
$1 = 20a-3b$

$$\begin{array}{r} 1 = 20a-3b \\ -(7 = 8a-3b) \\ \hline -6 = 12a \end{array}$$

$\boxed{a = -\frac{1}{2}}$

$7 = 8(-\frac{1}{2}) - 3b$

$7 = -4 - 3b$

$11 = -3b$

$\boxed{b = -11/3}$

Use the Intermediate Value Theorem to show that there is a real number a in $(0, 2)$ such that $f(a) = 12$ for $f(x) = -x^4 + 3x^3 + 5$.

$f(x)$ is a polynomial function and thus continuous on the interval $(0, 2)$.

$f(0) = -(0)^4 + 3(0)^3 + 5 = 5$

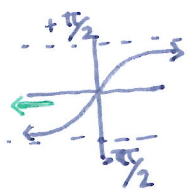
$f(2) = -(2)^4 + 3(2)^3 + 5 = -16 + 24 + 5 = 13$

As $f(0) < 12 < f(2)$, by the Intermediate Value theorem there exists a real number a in the interval $(0, 2)$ such that $f(a) = 12$.

2.6 Limits at Infinity and Horizontal Asymptotes

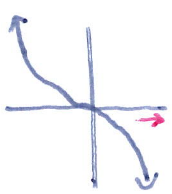
14. Compute each of the following limits.

(a) $\lim\limits_{x \to -\infty} (\arctan(x)) = -\dfrac{\pi}{2}$



(b) $\lim\limits_{x \to \infty} (6x^4 - 7x^9) = \lim\limits_{x \to \infty} \left(-7x^9 + 6x^4\right)$

$= -\infty$



15. Evaluate the limit, $\lim_{t\to\infty}\left(\frac{t^3-2t-3}{t^2-9}\right)$, if it exists. Then identify any horizontal asymptotes for the function.

$$\lim_{t\to\infty}\left(\frac{\overbrace{t^3-2t-3}^{\to\infty}}{\underbrace{t^2-9}_{\to\infty}}\right) = \lim_{t\to\infty}\left(\frac{(t^3-2t-3)}{(t^2-9)}\cdot\frac{(1/t^2)}{(1/t^2)}\right)$$

Multiply by 1 to "divide" by the dominant term of the denominator

$$= \lim_{t\to\infty}\left(\frac{\frac{t^3}{t^2}-\frac{2t}{t^2}-\frac{3}{t^2}}{\frac{t^2}{t^2}-\frac{9}{t^2}}\right)$$

$$= \lim_{t\to\infty}\left(\frac{t-\overbrace{2/t}^{\to 0}-\overbrace{3/t^2}^{\to 0}}{1-\underbrace{9/t^2}_{\to 0}}\right)$$

should always have a constant in denominator

$$= \infty$$

$$\lim_{t\to-\infty}\left(\frac{t^3-2t-3}{t^2-9}\right) = \lim_{t\to-\infty}\left(\frac{t-2/t-3/t^2}{1-9/t^2}\right)$$

$$= -\infty$$

No Horizontal asymptotes. always should check both limits

16. Evaluate the limit as x approaches $-\infty$ for $h(x)=\frac{2e^{3x}+e^{-2x}}{3e^{4x}+5e^{-2x}}$. Then identify any horizontal asymptotes for the function.

$$\lim_{x\to-\infty}\left(\frac{2e^{3x}+e^{-2x}}{3e^{4x}+5e^{-2x}}\right) = \lim_{x\to-\infty}\left(\frac{\overbrace{2e^{3x}}^{\to 0}+\overbrace{\frac{1}{e^{2x}}}^{\to\infty}}{\underbrace{3e^{4x}}_{\to 0}+\underbrace{\frac{5}{e^{2x}}}_{\to\infty}}\right)$$

$$= \lim_{x\to-\infty}\left[\left(\frac{2e^{3x}+1/e^{2x}}{3e^{4x}+5/e^{2x}}\right)\frac{(e^{2x})}{(e^{2x})}\right]$$

Multiply to eliminate the fractions because the fractions $\to\infty$

$$= \lim_{x\to-\infty}\left[\frac{\overbrace{2e^{5x}}^{\to 0}+1}{\underbrace{3e^{6x}}_{\to 0}+5}\right]$$

$$= \frac{1}{5}$$

$y=\frac{1}{5}$ is a horizontal asymptote

Note: $\lim_{x\to\infty} h(x) = 0$ so $h(x)$ has a horizontal asymptote at $y=0$

17. Identify any horizontal asymptotes for the function $f(x) = \dfrac{\sqrt{2x^2+3}+5x}{x+2}$ by evaluating the given limits.

$$\lim_{x\to\infty}\left(\frac{\overbrace{\sqrt{2x^2+3}+5x}^{\to\infty}}{\underbrace{x+2}_{\to\infty}}\right) = \lim_{x\to\infty}\left[\frac{\left(\sqrt{2x^2+3}+5x\right)}{(x+2)}\cdot\frac{(1/x)}{(1/x)}\right]$$

$$= \lim_{x\to\infty}\left[\frac{\dfrac{\sqrt{2x^2+3}}{x}+\dfrac{5x}{x}}{\dfrac{x}{x}+\dfrac{2}{x}}\right]$$

For $x > 0$ $\sqrt{x^2} = x$

$$= \lim_{x\to\infty}\left[\frac{\sqrt{\dfrac{2x^2}{x^2}+\dfrac{3}{x^2}}+5}{1+2/x}\right]$$

$$= \lim_{x\to\infty}\left[\frac{\overbrace{\sqrt{2+3/x^2}+5}^{\to\sqrt{2}+5}}{\underbrace{1+2/x}_{\to 1}}\right]$$

$$= \sqrt{2}+5$$

So the graph of $f(x)$ has a horizontal asymptote at $y=\sqrt{2}+5$

$$\lim_{x\to-\infty}\left(\frac{\sqrt{2x^2+3}+5x}{x+2}\right) = \lim_{x\to-\infty}\left[\frac{\left(\sqrt{2x^2+3}+5x\right)}{(x+2)}\cdot\frac{(-1/x)}{(-1/x)}\right]$$

For $x<0$
$\sqrt{x^2} = |x|$
and $\dfrac{-1}{x}$ is positive

$$= \lim_{x\to-\infty}\left[\frac{\sqrt{\dfrac{2x^2}{x^2}+\dfrac{3}{x^2}}+\dfrac{5x}{-x}}{\dfrac{x}{-x}+\dfrac{2}{-x}}\right]$$

$$= \lim_{x\to-\infty}\left[\frac{\overbrace{\sqrt{2+3/x^2}}^{\to\sqrt{2}}\;\overbrace{-\,5}^{\to -5}}{-1-\underbrace{2/x}_{\to 0}}\right]$$

$$= \frac{\sqrt{2}-5}{-1}$$

$$= -\sqrt{2}+5$$

So the graph of $f(x)$ has a horizontal asymptote at $y = -\sqrt{2}+5$