WIR#6



## Section 2.6 Part 2 - Related Rates

- In a Related Rates problem, we are trying to compute the rate of change of one quantity in terms of the rate of change of another quantity.
- **Related Rates Strategy**
  1. Assign variables to all quantities involved in the problem, and draw a picture, if applicable.
  2. State, in terms of the variables, the information that is given and the rate to be determined.
  3. Find an equation relating the variables introduced in step 1, if necessary.
  4. Differentiate both sides of the equation found in step 3 with respect to time. Remember to use the Chain Rule when taking the derivative of a variable that is a function of time, $t$.
  5. Substitute all known values into the equation from step 4, and then solve for the unknown rate of change.

①,②,③ 1. If $y = \sqrt[3]{3x^2+17}$, find $\frac{dx}{dt}$ if $\frac{dy}{dt} = 5$ when $x = 6$.

$y = (3x^2+17)^{1/3}$

④ $\frac{d}{dt}(y) = \frac{d}{dt}\left((3x^2+17)^{1/3}\right)$

$\frac{dy}{dt} = \frac{1}{3}(3x^2+17)^{-2/3} \cdot \left(3 \cdot 2x^1 \cdot \frac{dx}{dt} + 0\right)$

⑤ $5 = \frac{1}{3}(3(6)^2+17)^{-2/3} \cdot 6(6) \cdot \frac{dx}{dt}$

$5 = \frac{1}{3}(0.04)(36) \cdot \frac{dx}{dt}$

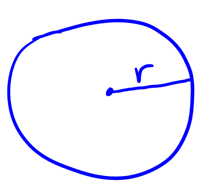
$5 = 0.48 \cdot \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{5}{0.48}$

$\boxed{\frac{dx}{dt} = \frac{125}{12}}$

2. Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 2 meters per second, how fast is the area of the spill increasing when the radius is 10 meters?

①




$r$ = radius
$A$ = area

② $\frac{dr}{dt} = 2$ m/s

$\frac{dA}{dt} = ?$

$r = 10$ m

③ $A = \pi r^2$

④ $\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$

$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

⑤ $\frac{dA}{dt} = 2\pi(10)(2) = \boxed{40\pi \text{ m}^2/\text{s}}$

3. A manufacturer determines the relationship between the demand and price for an item to be $2x^2 + 4xp + 7p^2 = 130{,}000$ where $x$ is the number of items demanded at a price of $\$p$ each. If the price per item is increasing at a rate of \$10.50 per month, find the rate of change of the demand with respect to time when the price is \$100 per item.

(2) $\frac{dp}{dt} = \$10.50/\text{month}$

$\frac{dx}{dt} = ?$

$p = \$100$

$x = 100$ items

We still need $x$:

$2x^2 + 4x(100) + 7(100)^2 = 130000$

$2x^2 + 400x + 70000 = 130000$

$-130000 \quad -130000$

$2x^2 + 400x - 60000 = 0$

$2(x^2 + 200x - 30000) = 0$

$2(x+300)(x-100) = 0$

$x + 300 = 0 \quad x - 100 = 0$

$\cancel{x = -300} \quad x = 100$

x must be non-negative!

(4) $\frac{d}{dt}(2x^2 + \underbrace{4x}_{F}\underbrace{p}_{S} + 7p^2) = \frac{d}{dt}(130000)$

$4x \cdot \frac{dx}{dt} + \left(4x \cdot \frac{dp}{dt} + p \cdot 4 \cdot \frac{dx}{dt}\right) + 14p \cdot \frac{dp}{dt} = 0$

(5) $4(100) \cdot \frac{dx}{dt} + 4(100)(10.5) + (100)(4)\frac{dx}{dt} + 14(100)(10.5) = 0$

$400 \cdot \frac{dx}{dt} + 4200 + 400 \cdot \frac{dx}{dt} + 14700 = 0$

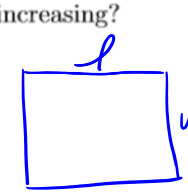
$800 \cdot \frac{dx}{dt} + 18900 = 0$

$\frac{800 \cdot \frac{dx}{dt}}{800} = \frac{-18900}{800}$

$\frac{dx}{dt} = \boxed{-23.625 \text{ items per month}}$

4. The length of a rectangle is increasing at a rate of 2 cm/s and its area is increasing at a rate of 20 cm$^2$/s. When the length is 3 cm and the area is 6 cm$^2$, how fast is the width of the rectangle increasing?

(1) A = area




(2) $\frac{d\ell}{dt} = 2\text{cm/s}$

$\frac{dA}{dt} = 20\text{cm}^2/\text{s}$

$\ell = 3\text{cm}$

$A = 6\text{cm}^2$

$\frac{dw}{dt} = ?$

$w = 2\text{cm}$

We still need $w$:

$A = \ell \cdot w$

$6 \quad 3$

$6 = 3 \cdot w$

$2 = w$

(3) $A = \ell \cdot w$

(4) $\frac{d}{dt}(A) = \frac{d}{dt}(\underbrace{\ell}_{F} \cdot \underbrace{w}_{S})$

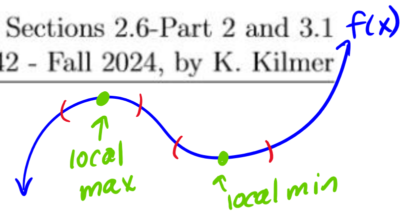
$\frac{dA}{dt} = \ell \cdot \frac{dw}{dt} + w \cdot \frac{d\ell}{dt}$

(5) $20 = 3\left(\frac{dw}{dt}\right) + (2)(2)$

$20 = 3 \cdot \frac{dw}{dt} + 4$

$16 = 3 \cdot \frac{dw}{dt}$

$\boxed{\frac{16}{3}\text{ cm/s} = \frac{dw}{dt}}$




## Section 3.1

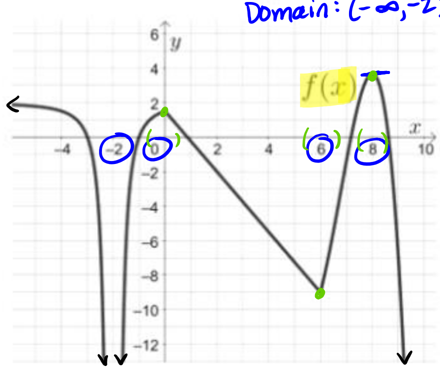
- If $f'(x) > 0$ on an interval, then $f$ is **increasing** on that interval.
- If $f'(x) < 0$ on an interval, then $f$ is **decreasing** on that interval.
- A **partition number of** $f'(x)$ is a value of $x$ such that $f'(x) = 0$ or $f'(x)$ is undefined. (These are the only places that $f'(x)$ might possibly change sign.)
- A **critical value** of $f(x)$ is a value of $x$ such that $x$ is a partition number of $f'(x)$ **and** $x$ is in the domain of $f(x)$. (These are the only places we might possibly have a local extremum.)
- The number $f(c)$ is a
  - **local maximum** if $f(c) \geq f(x)$ when $x$ is near $c$.
  - **local minimum** if $f(c) \leq f(x)$ when $x$ is near $c$.
  - **local extremum** if $f(c)$ is a local maximum or minimum.
- **The First Derivative Test:** Suppose that $x = c$ is a critical value of $f(x)$.
  - If $f'(x)$ changes from positive to negative at $x = c$, then $f(c)$ is a local maximum of $f(x)$.
  - If $f'(x)$ changes from negative to positive at $x = c$, then $f(c)$ is a local minimum of $f(x)$.
  - If the sign of $f'(x)$ is the same on both sides of $x = c$, then at $x = c$ there is neither a local max nor a local min.
- We will create a sign chart of $f'(x)$ to organize all of this information so that we can reach conclusions about $f(x)$. This process is summarized below:

**Step 1:** Determine the domain of $f$.
**Step 2:** Find the **partition numbers of** $f'$ (i.e. where $f'(x) = 0$ or $f'(x)$ does not exist).
**Step 3:** Determine which **partition numbers of** $f'$ are in the domain of $f$. These are the **critical values** of $f$.
**Step 4:** Create a sign chart of $f'(x)$ using the **partition numbers of** $f'$ found in step 2 to divide your sign chart into intervals. Then

- Select an $x$-value on each interval, evaluate $f'(x)$ at each $x$-value to determine the sign of $f'(x)$ and corresponding behavior of $f$ on each interval.
- Apply the **First Derivative Test** to find any local extrema of $f$.

5. Given the graph of $f(x)$ below, determine the following:

Domain: $(-\infty,-2)\cup(-2,\infty)$




(a) the partition numbers of $f'(x)$

$f'=0$ OR $f'$ DNE

$\boxed{x=8 \quad x=-2,\ x=0,\ x=6}$

(b) the critical values of $f(x)$

$\boxed{x=0,\ x=6,\ x=8}$

(c) the intervals on which $f(x)$ is increasing

$\boxed{(-2,0),\ (6,8)}$

(d) the intervals on which $f(x)$ is decreasing

$\boxed{(-\infty,-2),\ (0,6),\ (8,\infty)}$

(e) where the local extrema of $f(x)$ occur

Local max at $x=0$ ; $x=8$
Local min at $x=6$

6. For each function below, find the partition numbers of the derivative, the critical values of the function, the intervals on which the function is increasing, the intervals on which the function is decreasing, and find and classify all local extrema.

(a) $f(x) = \frac{1}{2}x^4 - 5x^3 + 3.5x^2 - 9$   Domain: $(-\infty, \infty)$

$f'(x) = 2x^3 - 15x^2 + 7x$
$= x(2x^2 - 15x + 7)$
$= x(2x-1)(x-7)$

$f'(x) = 0$ when $x(2x-1)(x-7) = 0$
$x = 0$   $2x - 1 = 0$   $x - 7 = 0$
$2x = 1$   $x = 7$
$x = 1/2$

$f'(x)$ is always defined since it is a polynomial!

$f$: ↘ ↗ ↘ ↗
$f'$: $-$ | $+$ | $-$ | $+$
$x=-1$, $0$ CV, $x=\frac{1}{4}$, $1/2$ CV, $x=1$, $7$ CV, $x=8$

- The partition numbers of $f'$ are $x=0, x=\frac{1}{2}, x=7$.
- The critical values of $f$ are $x=0, x=1/2, x=7$.
- $f$ is increasing on $(0, 1/2), (7, \infty)$.
- $f$ is decreasing on $(-\infty, 0), (1/2, 7)$.
- Local minimum of $-9 \leftarrow f(0)$ at $x=0$.
- Local maximum of $\frac{-279}{32} \leftarrow f(1/2)$ at $x = 1/2$.
- Local minimum of $-352$ at $x=7$.
  $f(7)$

(b) $g(x) = \dfrac{2x^2 + 40}{x - 4}$   Domain: $x - 4 \neq 0$, $x \neq 4$

$g'(x) = \dfrac{(x-4)(4x) - (2x^2+40)(1)}{(x-4)^2}$

$= \dfrac{4x^2 - 16x - 2x^2 - 40}{(x-4)^2}$

$= \dfrac{2x^2 - 16x - 40}{(x-4)^2} \leftarrow y_1$

$g'(x) = 0$ when $2x^2 - 16x - 40 = 0$
$2(x^2 - 8x - 20) = 0$
$2(x+2)(x-10) = 0$
$x + 2 = 0$   $x - 10 = 0$
$x = -2$   $x = 10$

$g'(x)$ DNE   $(x-4)^2 = 0$
$x - 4 = 0$
$x = 4$

partition #'s of $g'$

$g$: ↗ ↘ ↘ ↗
$g'$: $+$ | $-$ ⋮ $-$ | $+$
$x=-3$, $-2$ CV, $x=0$, $4$, $x=5$, $10$ CV, $x=11$

- The partition numbers of $g'$ are $x=-2, x=4, x=10$.
- The critical values of $g$ are $x=-2, x=10$.
- $g$ is increasing on $(-\infty, -2), (10, \infty)$.
- $g$ is decreasing on $(-2, 4), (4, 10)$.
- Local maximum of $-8 \leftarrow g(-2)$ at $x = -2$.
- Local minimum of $40 \leftarrow g(10)$ at $x = 10$.

(c) $C(x) = \sqrt[3]{(4x^2-3)^2}$

odd root ⇒ No Domain restrictions!

Domain: $(-\infty, \infty)$

$C(x) = \left((4x^2-3)^2\right)^{1/3}$

$C(x) = (4x^2-3)^{2/3}$

$C'(x) = \frac{2}{3}(4x^2-3)^{-1/3} \cdot 8x$

$= \frac{2}{3} \cdot \frac{1}{(4x^2-3)^{1/3}} \cdot 8x$

$= \frac{16x}{3(4x^2-3)^{1/3}}$ $y_1$

$C$: ↘ ↗ ↘ ↗

$C'$: $-$ $+$ $-$ $+$

$x=-1$ | $-\frac{\sqrt{3}}{2} \approx -.87$ CV | $x=-1/2$ | $0$ CV | $x=1/2$ | $\frac{\sqrt{3}}{2} \approx .87$ CV | $x=1$

$C'(x)=0$ when $16x=0$
$x=0$

partition #'s of $C'$

$C'(x)$ DNE when $3(4x^2-3)^{1/3}=0$

$\left((4x^2-3)^{1/3}\right)^3 = (0)^3$

$4x^2-3=0$

$4x^2=3$

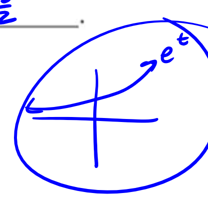
$\sqrt{x^2} = \sqrt{\frac{3}{4}}$

$x = \frac{\pm\sqrt{3}}{2}$

- The partition numbers of $C'$ are $x=-\frac{\sqrt{3}}{2}, x=0, x=\frac{\sqrt{3}}{2}$.
- The critical values of $C$ are $x=-\frac{\sqrt{3}}{2}, x=0, x=\frac{\sqrt{3}}{2}$.
- $C$ is increasing on $\left(-\frac{\sqrt{3}}{2}, 0\right), \left(\frac{\sqrt{3}}{2}, \infty\right)$.
- $C$ is decreasing on $\left(-\infty, -\frac{\sqrt{3}}{2}\right), \left(0, \frac{\sqrt{3}}{2}\right)$.
- Local minimum of $0 \leftarrow C\left(-\frac{\sqrt{3}}{2}\right)$ at $x=-\frac{\sqrt{3}}{2}$.
- Local maximum of $2.0801 \leftarrow C(0)$ at $x=0$.
- Local minimum of $0 \leftarrow C\left(\frac{\sqrt{3}}{2}\right)$ at $x=\frac{\sqrt{3}}{2}$.

(d) $f(t) = t^8e^{-t} = \frac{t^8}{e^t}$

Domain: $(-\infty, \infty)$




$f'(t) = \frac{e^t(8t^7) - t^8(e^t)}{(e^t)^2} = \frac{\cancel{e^t}(8t^7-t^8)}{\cancel{e^t} \cdot e^t}$

$= \frac{t^7(8-t)}{e^t}$ $y_1$

$f$: ↘ ↗ ↘

$f'$: $-$ $+$ $-$

$t=-1$ | $0$ CV | $t=1$ | $8$ CV | $t=9$

$f'(t)=0$ when $t^7(8-t)=0$

$t^7=0$ $\quad 8-t=0$

$t=0$ $\quad t=8$

$f'(t)$ DNE when $e^t=0$

$e^t$ is never zero!

- The partition numbers of $f'$ are $t=0$ & $t=8$.
- The critical values of $f$ are $t=0$ & $t=8$.
- $f$ is increasing on $(0, \infty)$.
- $f$ is decreasing on $(-\infty, 0), (8, \infty)$.
- Local minimum of $0 \leftarrow f(0)$ at $t=0$.
- Local maximum of $\frac{16777216}{e^8} \leftarrow f(8)$ at $t=8$.

7. Given $f'(x) = \dfrac{n(x)(x+3)^4(x-5)^3}{(x+6)^3}$, the domain of $f(x)$ is $(-\infty,-6)\cup(-6,\infty)$, and $n(x)$ is a function that is always negative, find the partition numbers of $f'(x)$, the critical values of $f(x)$, the intervals on which $f(x)$ is increasing, the intervals on which $f(x)$ is decreasing, and the location of all local extrema and classify.

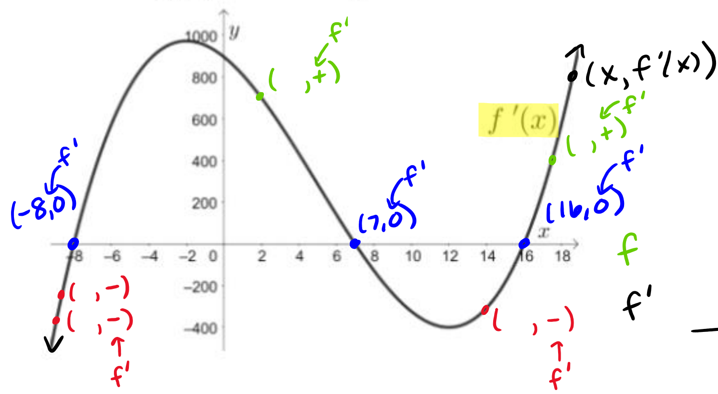
$f'(x)=0$ when $x=-3, x=5$
$f'(x)$ DNE when $x=-6$

f: ↘ ↗ ↗ ↘
f': − + + −
$x=-7$ | $-6$ | $x=-5$ | $-3$ CV | $x=0$ | $5$ CV | $x=6$

Not a min since $x=-6$ is not a CV (i.e. not in domain of f)

- The partition numbers of $f'$ are $x=-6, x=-3, x=5$.
- The critical values of $f$ are $x=-3, x=5$.
- $f$ is increasing on $(-6,-3), (-3,5)$.
- $f$ is decreasing on $(-\infty,-6), (5,\infty)$.
- Local maximum at $x=5$.

8. Given the graph of $f'(x)$ below and that the domain of $f(x)$ is $(-\infty,\infty)$, determine the partition numbers of $f'(x)$, the critical values of $f(x)$, the intervals on which $f(x)$ is increasing, the intervals on which $f(x)$ is decreasing, and where the local extrema of $f(x)$ occur.




f: ↘ ↗ ↘ ↗
f': − + − +
$-8$ CV | $7$ CV | $16$ CV

(f′=0 or f′ DNE) Note: f′ always exists

- The partition numbers of $f'$ are $x=-8, x=7, x=16$.
- The critical values of $f$ are $x=-8, x=7, x=16$.
- $f$ is increasing on $(-8,7), (16,\infty)$.
- $f$ is decreasing on $(-\infty,-8), (7,16)$.
- Local maximum at $x=7$.
- Local minimum at $x=-8, x=16$.

Domain: $\mathbb{R}$

9. The revenue function for a particular TV is given by $R(x) = -2x^2 + 1000x$ (in dollars) for $0 \leq x \leq 500$ when $x$ TVs are sold. Determine the intervals where $R(x)$ is increasing/decreasing and the local extrema of $R(x)$.

$R'(x) = -4x + 1000$

$R'(x) = 0$ when $-4x + 1000 = 0$

$1000 = 4x$

$250 = x$

R: ↗ ↘

R': + −

0 — x=100 — 250 — x=300 — 500

$$\boxed{\begin{array}{l} R(x) \text{ is increasing on } (0, 250) \\ R(x) \text{ is decreasing on } (250, 500) \\ \text{Local max } (250 \text{ TVs}, \$375{,}000) \end{array}}$$

R(250)