1.2: SOLUTIONS TO DIFFERENTIAL EQUATIONS

Review

- A **solution to a differential equation** is

  a function that satisfies the equation when plugged in.

- An **initial value problem** is

  a differential equation plus an initial condition.

- A **solution to an initial value problem** is

  a solution to the differential equation that also satisfies the initial condition.

Exercise 1

Is e^{2x} a solution to the differential equation $y'' - 4y' + 4y = 0$?

Plug in $y(x) = e^{2x}$:

$$y(x) = e^{2x}$$
$$y'(x) = 2e^{2x}$$
$$y''(x) = 4e^{2x}$$

$$y'' - 4y' + 4y = 0$$
$$4e^{2x} - 4(2e^{2x}) + 4(e^{2x}) = 0$$
$$4e^{2x} - 8e^{2x} + 4e^{2x} = 0 \checkmark$$

$\boxed{\text{Yes, } e^{2x} \text{ is a solution.}}$

Exercise 2

Is $\cos(x)$ a solution to the differential equation $f^{(4)}(x) - f''(x) = 4\cos(x)$?

$$f(x) = \cos(x)$$
$$f' = -\sin(x)$$
$$f'' = -\cos(x)$$
$$f''' = \sin(x)$$
$$f^{(4)} = \cos(x)$$

$$f^{(4)} - f'' = 4\cos(x)$$
$$\cos(x) - (-\cos(x)) = 4\cos(x)$$
$$2\cos(x) = 4\cos(x) \times$$

$\boxed{\text{No, } \cos(x) \text{ is not a solution.}}$

Exercise 3

Is $\sin(2t)$ a solution to the following initial value problem?

$$\frac{d^2g}{dt^2} - \frac{dg}{dt} + 4g = -2\cos(2t), \qquad g(0) = 1.$$

$g(t) = \sin(2t)$

$g'(t) = 2\cos(2t)$

$g''(t) = -4\sin(2t)$

$\cancel{-4\sin(2t)} - 2\cos(2t) + \cancel{4(\sin(2t))} = -2\cos(2t)$ ✓

So, $\sin(2t)$ satisfies the diff equation.

But does it also satisfy the initial condition?

$g(0) = \sin(2\cdot 0) = 0 \neq 1$ ✗

No. So, $\sin(2t)$ is not a solution to the IVP.

Exercise 4

Find the values of a for which e^{at} is a solution to $y'' - 3y' + y = 0$.

$y = e^{at}$

$y' = ae^{at}$

$y'' = a^2e^{at}$

$a^2e^{at} - 3ae^{at} + e^{at} = 0$

$(a^2 - 3a + 1)e^{at} = 0$

$a^2 - 3a + 1 = 0$

$a = \dfrac{-(-3) \pm \sqrt{9 - 4(1)(1)}}{2} = \boxed{\dfrac{3 \pm \sqrt{5}}{2}}$

Exercise 5

Find the values of b such that $\sin(bx)$ solves the differential equation $y + 6y'' = 0$.

$$y = \sin(bx)$$

$$y' = b\cos(bx)$$

$$y'' = -b^2 \sin(bx)$$

$$y + 6y'' = 0$$

$$\sin(bx) + 6\left(-b^2 \sin(bx)\right) = 0$$

$$(1 - 6b^2)\sin(bx) = 0$$

Either $1 - 6b^2 = 0$ or $\sin(bx) = 0$ for all x.

$$b^2 = \frac{1}{6} \qquad\qquad b = 0$$

$$b = \pm \frac{1}{\sqrt{6}}$$

1.3: CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Review

- An **ordinary differential equation** (ODE) is a differential equation that has derivatives with respect to

  just one variable.

- A **partial differential equation** (PDE) is a differential equation that has derivatives with respect to

  more than one variable.

- The **order** of a differential equation is

  the order of the highest derivative.

- An ODE is **linear** if it can be written in the form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x).$$

  i.e., in a **linear** ODE,

  - y and its derivatives are all in separate terms.
  - y and its derivatives are not inside any functions or to any powers.
  - Each term can be multiplied by a function of x.
  - There can also be another function of x by itself.

Exercise 6

For each of the following, determine whether it is an ODE or a PDE. Additionally, state the order of the differential equation.

(a) $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} = f$

has x and y derivatives $\Rightarrow$ PDE   1st order

(b) $\dfrac{\mathrm{d}^2 g}{\mathrm{d}x^2} - 2\left(\dfrac{\mathrm{d}g}{\mathrm{d}x}\right)^5 = xg$

only x derivatives $\Rightarrow$ ODE   2nd order

(c) $r'''(z)r'(z) - z^2 + \tan(z)r(z) = 0$

only z derivatives $\Rightarrow$ ODE   3rd order

(d) $u_{xx} + u_{yy}u = 0$

x and y derivatives $\Rightarrow$ PDE   2nd order

Exercise 7

For each of the following ODEs, determine if it is linear or nonlinear.

(a) $w' - w''w + t^2 w = 7t$

nonlinear

(b) $\dfrac{1}{g'(t)} + g(t) = g''(t)$

nonlinear

(c) $(x^2 + \cos(x))Q(x) - \tan(x)Q'(x) = Q'''(x)$

linear

(d) $y^{(5)} - x^3 y^2 + y''' = 7x^3 - \csc(x)$

nonlinear

(e) $t^2 + z^{(6)} + \cos(t)z''' = \cos(t)$

linear

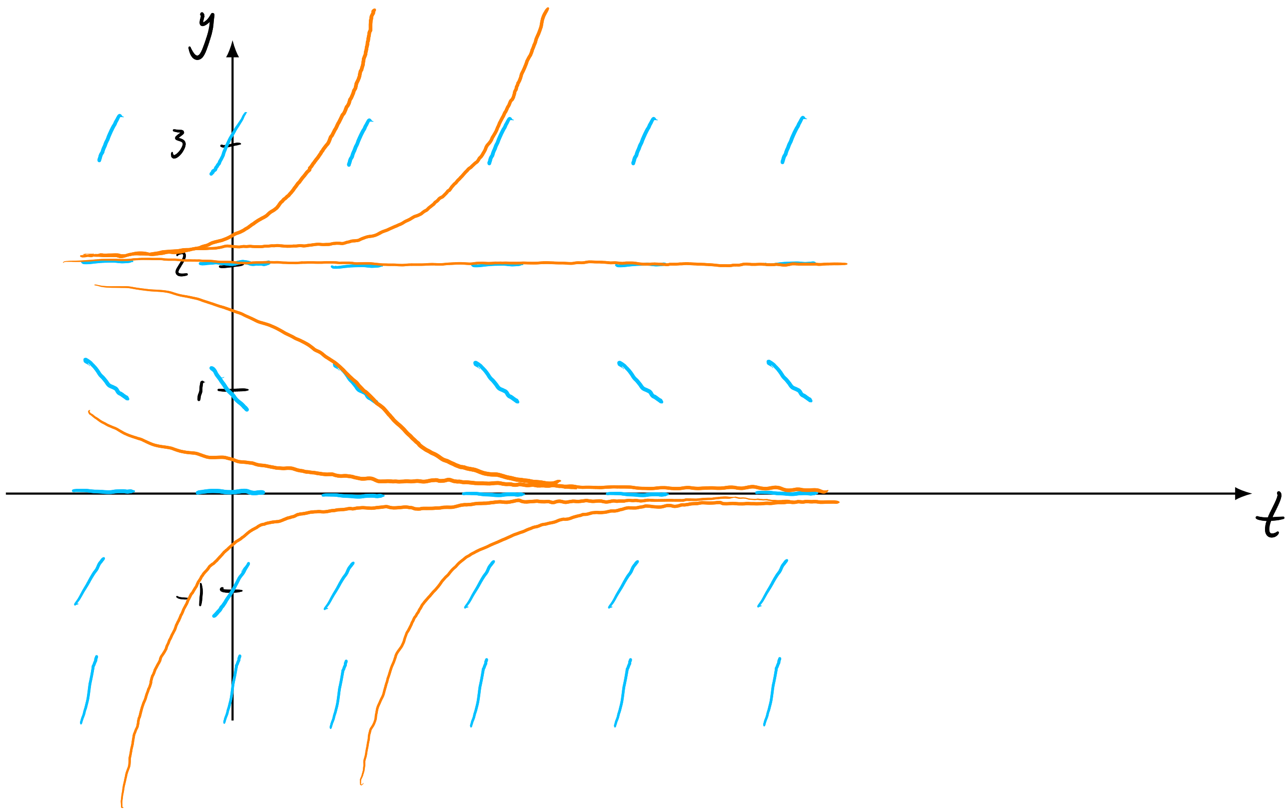
1.1: DIRECTION FIELDS

Review

- A **direction field** (or **slope field**) plots the slope of the solution to an ODE at a bunch of different points.

Exercise 8

Sketch the slope field of for the differential equation $y' = y^2 - 2y$. Draw some example solutions to the ODE. If the initial condition is $y(0) = a$, how does the long-time behavior of $y(t)$ depend on a?



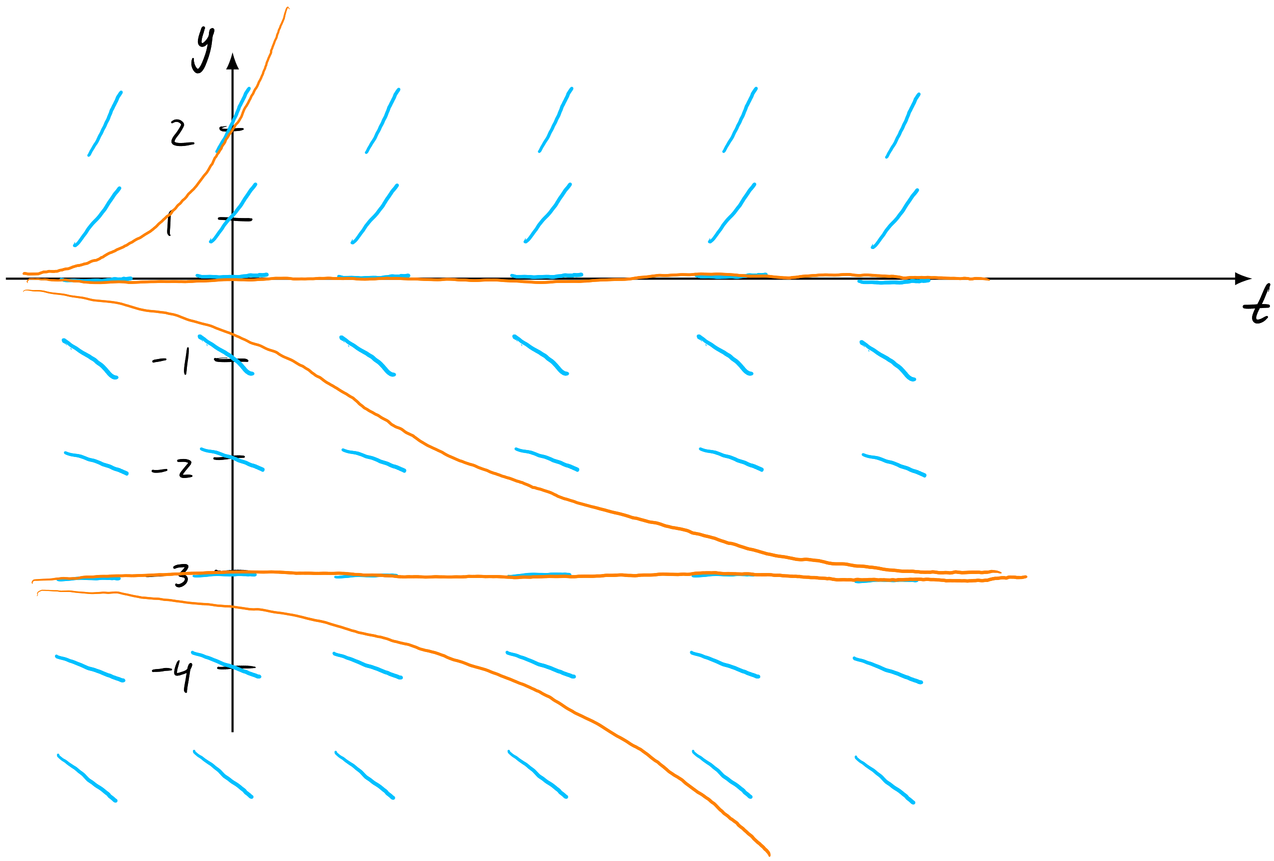
If $a < 2$, then $\lim_{t\to\infty} y(t) = 0$.

If $a = 2$, then $\lim_{t\to\infty} y(t) = 2$.

If $a > 2$, then $\lim_{t\to\infty} y(t) = \infty$.

Exercise 9

Sketch the slope field of for the differential equation $y' = \frac{1}{4}y(y+3)^2$. Draw some example solutions to the ODE. If the initial condition is $y(0) = a$, how does the long-time behavior of $y(t)$ depend on a?



If $a < -3$, then $\lim_{t\to\infty} y(t) = -\infty$.

If $-3 \le a < 0$, then $\lim_{t\to\infty} y(t) = -3$.

If $a = 0$, then $\lim_{t\to\infty} y(t) = 0$.

If $a > 0$, then $\lim_{t\to\infty} y(t) = \infty$.

2.2: SEPARABLE ODES – SEPARATION OF VARIABLES

Review

- A **separable** ODE is an ODE that has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y).$$

- Steps for solving a separable ODE:
  1. Treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a fraction.
  2. Move all the y's to one side and all the x's to the other.
  3. Integrate both sides.
  4. (If possible) solve for y.

- The **general solution** to a differential equation is the form of the solution that contains all possible solutions inside it. It is the solution you get *before* you plug in the initial condition to solve for c.

- The solution to an initial value problem is **defined** on an *interval* that contains the initial condition. On that interval, the solution must be
  - a **function** that is
  - **defined** and
  - **differentiable**.

Exercise 10

Solve the differential equation $f' = \frac{x^3+1}{f^2}$.

$$\frac{df}{dx} = \frac{x^3+1}{f^2}$$

$$\int f^2\,df = \int (x^3+1)\,dx$$

$$\frac{1}{3}f^3 = \frac{1}{4}x^4 + x + c$$

$$\boxed{f(x) = \left(\frac{3}{4}x^4 + 3x + c\right)^{1/3}}$$

this is a solution for any value of c.

Exercise 11

Solve the initial value problem

$$f' = e^{-f}(4-2x), \qquad f(2) = 0.$$

Where is the solution defined?

$$\frac{df}{dx} = e^{-f}(4-2x)$$

$$\int e^{f}\,df = \int (4-2x)\,dx$$

$$e^{f} = 4x - x^2 + c$$

$$f(x) = \ln(4x - x^2 + c) \leftarrow \text{general solution}$$

Use IC to solve for c:

$$f(2) = \ln\left(4(2) - (2)^2 + c\right) = 0$$

$$\ln(4 + c) = 0$$

$$4 + c = e^0 = 1$$

$$c = -3$$

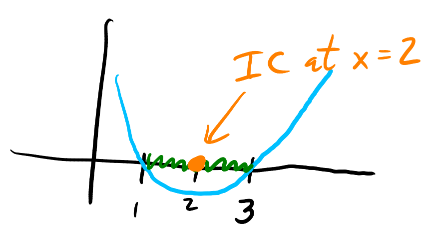
$$\boxed{f(x) = \ln(4x - x^2 - 3)}$$

Where is the solution defined?

$$-x^2 + 4x - 3 > 0$$

$$x^2 - 4x + 3 < 0$$

$$(x-1)(x-3) < 0$$



$$\boxed{\text{Solution is defined on } (1, 3).}$$

Exercise 12

Solve the initial value problem

$$(e^y - y)x^2y' = 1, \qquad y(1) = 2.$$

$$(e^y - y)\,x^2 \frac{dy}{dx} = 1$$

$$\int (e^y - y)\,dy = \int x^{-2}\,dx$$

$$e^y - \tfrac{1}{2}y^2 = -x^{-1} + c$$ ← general solution in implicit form. We leave it in implicit form since we can't solve for y.

Use IC to find c:

$$e^2 - \tfrac{1}{2}(2)^2 = -1^{-1} + c$$

$$e^2 - 2 = -1 + c$$

$$c = e^2 - 1$$

$$\boxed{e^y - \tfrac{1}{2}y^2 = -x^{-1} + e^2 - 1}$$ solution to the IVP in implicit form.

Exercise 13

(a) Find the general solution to the differential equation $y' + y^2 \sin(x) = 0$.

(b) Find the solution that satisfies the initial condition $y(\pi) = 3$. Where is the solution defined?

(c) Find the solution that satisfies the initial condition $y(\pi) = 0$. Where is the solution defined?

a) $\dfrac{dy}{dx} = -y^2 \sin(x)$

$\int y^{-2}\,dy = -\int \sin(x)\,dx$ $(y \neq 0)$

$-y^{-1} = \cos(x) + C$

$y^{-1} = C - \cos(x)$

$y(x) = \dfrac{1}{C - \cos(x)}$

Case $y = 0$:

$\Rightarrow y'(x) = 0.$

plug into diff eq:

$0 = -(0)^2 \sin(x)$

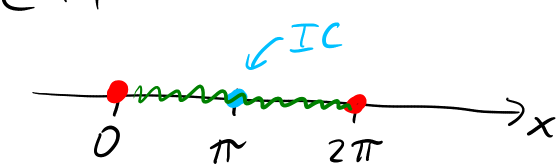
$0 = 0$ ✓

$y = 0$ is a solution

General solution: $\boxed{y(x) = \dfrac{1}{C - \cos(x)} \quad \text{or} \quad y = 0}$

b) $y(0) = \dfrac{1}{C - \cos(\pi)} = \dfrac{1}{C+1} = \dfrac{1}{2} \Rightarrow C + 1 = 2 \Rightarrow C = 1$

$\boxed{y(x) = \dfrac{1}{1 - \cos(x)}}$



defined on $(0, 2\pi)$.

c) $\boxed{y(x) = 0}$ define on $(-\infty, \infty)$.

Exercise 14

Solve the differential equation $\frac{\mathrm{d}g}{\mathrm{d}t} = (g^2 - 9)\cos(t)$.

$$\int \frac{1}{g^2-9}\,dg = \int \cos(t)\,dt \quad (g^2 \neq 9)$$

$$\frac{1}{g^2-9} = \frac{1}{(g-3)(g+3)} = \frac{A}{g-3} + \frac{B}{g+3}$$

$$1 = A(g+3) + B(g-3)$$

$g=3$: $1 = 6A \Rightarrow A = 1/6$

$g=-3$: $1 = -6B \Rightarrow B = -1/6$

$$\int\left(\frac{1/6}{g-3} - \frac{1/6}{g+3}\right)dg = \sin(t) + C$$

$$\boxed{\frac{1}{6}\ln|g-3| - \frac{1}{6}\ln|g+3| = \sin(t) + C \quad \text{or } g(x) = 3 \quad \text{or } g(x) = -3}$$

It is possible to solve explicitly for $g(t)$, but I'm just going to leave it in implicit form.

case $g=3$:

$g(x) = 3$, $g'(x) = 0$

plug into diff eq:

$0 = (3^2 - 9)\cos(x) = 0$ ✓

$g(x) = 3$ is a solution.

Case $g=-3$:

$g(x) = -3$, $g'(x) = 0$

plug into diff eq:

$0 = ((-3)^2 - 9)\cos(x) = 0$ ✓

$g(x) = -3$ is a solution.

2.1: LINEAR ODES – METHOD OF INTEGRATING FACTORS

Review

Whenever you have a **linear equation**, you can always solve it using the **method of integrating factors**.

Steps for the **method of integrating factors**:

1. Put in standard form: $y' + p(t)y = g(t)$.
2. Multiply by μ.
3. Find μ to match the product rule.
4. Reverse the product rule.
5. Integrate both sides and solve for y.

Exercise 15

Determine if each of the following are separable or linear.

(a) $u'(t) = \dfrac{\sin(t)}{\cos(u)}$

separable

(b) $\dfrac{\mathrm{d}w}{\mathrm{d}r} = \sin(wr)$

neither

(c) $xz^2\dfrac{\mathrm{d}z}{\mathrm{d}x} = 1$ $\Rightarrow$ $\dfrac{dz}{dx} = \dfrac{1}{xz^2}$

separable

(d) $y' = 3y + 4$

separable and linear

(e) $\dfrac{\mathrm{d}g}{\mathrm{d}t} = 4g - 3t$

linear

(f) $t^2y - y' = 2$ $\Rightarrow$ $y' = t^2y - 2$

linear

(g) $f' = 1 + t + f + tf$ $= 1+t + (1+t)f = (1+t)(1+f)$

separable and linear

Exercise 16

Solve the differential equation $y' = 3y + 4$. (Note that this could also be solved using separation of variables.)

1. Put in standard form.

$$y' - 3y = 4$$

2. Multiply by μ.

$$\mu y' \underbrace{- 3\mu}_{\frac{d\mu}{dt}} y = 4\mu$$

3. Find μ.

$$\frac{d\mu}{dt} = -3\mu \Rightarrow \mu(t) = e^{-3t}$$

4. Reverse product rule.

$$\frac{d}{dt}\left(e^{-3t} y\right) = 4e^{-3t}$$

5. Integrate and solve for y.

$$e^{-3t} y = \frac{-4}{3} e^{-3t} + c$$

$$\boxed{y(t) = \frac{-4}{3} + ce^{3t}}$$

Exercise 17

Solve the initial value problem

$$tf' - (1+t)f = 2t^2, \qquad f(0) = 2.$$

1. Put in standard form.

$$f' - \left(\frac{1}{t} + 1\right) f = 2t$$

2. Multiply by μ.

$$\mu(t) f' \underbrace{- \left(\frac{1}{t} + 1\right)\mu(t)}_{\frac{d\mu}{dt}} f = 2t\,\mu(t)$$

3. Find $\mu(t)$.

$$\frac{d\mu}{dt} = -\left(\frac{1}{t} + 1\right)\mu$$

$$\int \frac{d\mu}{\mu} = -\int \left(\frac{1}{t} + 1\right) dt$$

$$\ln|\mu| = -\ln|t| - t + c$$

$$\mu(t) = ce^{-\ln|t| - t} = ce^{\ln\left|\frac{1}{t}\right|} e^{-t} = c\left|\frac{1}{t}\right| e^{-t} = \cancel{c}\,1\,\frac{e^{-t}}{t}$$

4. Reverse the product rule.

$$\frac{d}{dt}\left(\frac{e^{-t}}{t} f(t)\right) = 2t\,\frac{e^{-t}}{t} = 2e^{-t}$$

5. Integrate both sides and solve for f.

$$\frac{e^{-t}}{t} f(t) = -2e^{-t} + c$$

$$f(t) = -2t + cte^{t}$$

6. Use IC to solve for c.

$$2 = -2(1) + c(1)e^{1}$$

$$4 = ec$$

$$\Rightarrow c = \frac{4}{e}$$

$$\boxed{f(t) = -2t + \frac{4}{e}te^{t}}$$