Week in Review
Math 152

Week 13

Test 3 Review II

Common Exam III Prep. Part II

1. Which of the following statements is true about the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$?

   (a) The series diverges by the Test of Divergence.

   (b) The series diverges by the Comparison Test by comparison to $\sum_{n=1}^{\infty} \frac{1}{n}$.

   (c) The series converges by the Limit Comparison Test by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

   (d) The series converges by the Comparison Test by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

   (e) The series diverges by the Limit Comparison Test by comparison to $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{x\to 0} \frac{\sin(x)}{x} = 1 \text{ where } \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \text{ (p series)}$$

Substitute $x = \frac{1}{n^2}$

Common Exam III Prep. Part II

Using the Alternating Series Estimation Theorem, find the error R_5 when using the $5th$ partial sum, s_5, to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{(-3)^n}{(2n)!}$.

(a) $|R_5| \leq \frac{3^6}{(2)!(6)!}$

(b) $|R_5| \leq \frac{3^6}{(12)!}$

(c) $|R_5| \leq \frac{3^5}{(10)!}$

(d) $|R_5| \leq -\frac{3^6}{(12)!}$

(e) $|R_5| \leq -\frac{3^5}{(2)!(5)!}$

$$|S - S_5| \leq |a_6| = \frac{3^6}{12!}$$

Common Exam III Prep. Part II

For which of the following series will the Ratio test be inconclusive?

(a) $\sum_{n=1}^{\infty} \frac{n}{(-2)^n}$

(b) $\sum_{n=1}^{\infty} n^2 e^{-n}$

(c) $\sum_{n=1}^{\infty} \frac{n3^n}{2^n(n+1)!}$

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{\ln n}}$

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

(d) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\sqrt{\ln(n+1)}}{\sqrt{\ln(n)}}\right| = 1$

Common Exam III Prep. Part II

Which of the following statements is true for the following series?

(I) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (II) $\sum_{n=1}^{\infty} (-1)^n e^{-2n}$ (III) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n+1)!}$

(a) (I) converges, but not absolutely; (II) converges absolutely; (III) diverges.

(b) (I), (II) and (III) converge absolutely.

(c) (I) diverges, (II) converges, but not absolutely; (III) converges absolutely.

(d) (I) converges, but not absolutely; (II) and (III) diverge.

(e) (I) converges, but not absolutely; (II) and (III) converge absolutely.

(I) $\sum \frac{(-1)^n}{n} < \infty$ (Alternating Harmonic)

(II) $\sum (-1)^n e^{-2n} < \infty$ ($e^{-2n} \to 0$)

(III) $\sum \frac{(-1)^n n!}{(2n+1)!} < \infty$ ($\frac{n!}{(2n+1)!} \to 0$)

(I) $\sum \frac{1}{n} = \infty$ (Harmonic)

(II) $\sum e^{-2n} < \infty$ (Geometric)

(III) $\sum \frac{n!}{(2n+1)!} = \sum \frac{1}{(2n+1)(2n)\cdots(n+1)}$

$< \sum \frac{1}{(n+1)^2} < \infty$

Common Exam III Prep. Part II

Find all value(s) of p for which the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges.

(a) $p \geq 1$.

(b) $p > 0$.

(c) $p \geq 0$.

(d) $p > 1$.

(e) This series will always diverge.

If $p > 0$, $\lim_{n\to\infty} \frac{1}{n^p} \to 0$

Common Exam III Prep. Part II

Find the MacLaurin series for $f(x) = \dfrac{x^3}{4+x}$.

(a) $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^{n+3}}{4^{n+1}}, \quad |x| < 4$

(b) $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{4^{n+1}}, \quad |x| < 4$

(c) $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{4^{n+1}}, \quad |x| < \frac{1}{4}$

(d) $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^{n+3}}{4^{n+1}}, \quad |x| < \frac{1}{4}$

(e) $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{4^{n+1}}, \quad |x| < 4$ ← correct

$$\frac{x^3}{4}\left(\frac{1}{1+\frac{x}{4}}\right)$$

$$= \frac{x^3}{4} \sum_{n=0}^{\infty} \left(-\frac{x}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+3}}{4^{n+1}}$$

$$\left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$$

Common Exam III Prep. Part II

Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(-2)^n x^2}{(n+1)!}$.

(a) 0

(b) ∞ $\leftarrow$ correct

(c) $\frac{1}{2}$

(d) 1

(e) 2

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}x^2}{(n+2)!}\cdot\frac{(n+1)!}{(-2)^n x^2}\right|$$

$$= \lim_{n\to\infty}\frac{2}{(n+2)} = 0 \text{ for all } x$$

Common Exam III Prep. Part II

Find the 23$^{\text{rd}}$ derivative at $x = 3$, $f^{(23)}(3)$, for $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}(x-3)^n}{(n+1)^2}$.

(a) $f^{(23)}(3) = \frac{2^{24}}{(24)^2}$

(b) $f^{(23)}(3) = -\frac{2^{24}}{(24)^2}$

(c) $f^{(23)}(3) = -\frac{2^{24}(23)!}{(24)^2}$ ← correct

(d) $f^{(23)}(3) = \frac{2^{24}(23)!}{(24)^2}$

(e) $f^{(23)}(3) = -\frac{2^{24}}{(24)^2(23)!}$

$$\sum \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$n = 23$$

$$\frac{f^{(23)}(3)}{23!} = -\frac{2^{24}}{24^2}$$

$$f^{(23)}(3) = -\frac{2^{24}(23!)}{24^2}$$

Common Exam III Prep. Part II

Using a MacLaurin series, evaluate the integral $\int_0^2 e^{-x^2}\,dx$.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)n!}$ ← correct

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{n!}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{n!}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+2}}{(2n+2)n!}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{(n+1)n!}$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

$$\int_0^2 e^{-x^2} dx = \sum_{n=0}^{\infty} \int_0^2 \frac{(-1)^n}{n!} x^{2n} dx$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{n!(2n+1)} x^{2n+1}\right]_0^2$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} 2^{2n+1}$$

Common Exam III Prep. Part II

Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

(a) 1

(b) $\frac{1}{4}$

(c) $-\frac{1}{4}$

(d) $-\frac{\pi}{4}$

(e) $\frac{\pi}{4}$ ← correct

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\int_0^x \frac{d}{dx}(\tan^{-1} x)dx = \int_0^x \sum_{n=0}^{\infty}(-1)^n x^{2n}\, dx$$

$$\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\tan^{-1} 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Common Exam III Prep. Part II

Find a 3rd degree Taylor Polynomial for a function $f(x)$ using the following information.

$$f(1) = 1, \qquad f'(1) = \frac{1}{2}, \qquad f''(1) = -\frac{1}{4}, \qquad f'''(1) = \frac{3}{8}$$

(a) $T(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$

(b) $T(x) = 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{3}{8}x^3$

(c) $T(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

(d) $T(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{3}{8}(x-1)^3$

(e) $T(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$ $\leftarrow$ correct

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3$$
$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$$

Common Exam III Prep. Part II

Which of the following series is absolutely convergent by **the Ratio Test**?

(I) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$ (II) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4(-2)^n}{n!}$ (III) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^3+4}$

(a) I and II only ← correct

(b) I only

(c) II only

(d) II and III only

(e) I, II, and III

$$\text{(I)} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3}{3^{n+1}}\cdot\frac{3^n}{n^3}\right| = \lim_{n\to\infty}\left|\frac{1}{3}\cdot\left(\frac{n+1}{n}\right)^3\right| = \frac{1}{3} < 1$$

$$\text{(II)} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^4(-2)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^4(-2)^n}\right| = \lim_{n\to\infty}\left|\frac{2}{(n+1)}\left(\frac{n+1}{n}\right)^4\right| = 0$$

$$\text{(III)} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)}{(n+1)^3+4}\cdot\frac{n^3+4}{(-1)^n n}\right| = 1$$

Common Exam III Prep. Part II

Which of the following is correct MacLaurin series for $f(x) = x\cos(\sqrt{x})$?

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(2n)!}$ $\leftarrow$ correct

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(2n+1)!}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\cos\sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{n}$$

$$x\cos\sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{n+1}$$

Common Exam III Prep. Part II

Find the sum of the series $\sum_{n=2}^{\infty} \frac{3^n}{n!}$.

(a) e^3

(b) $e^3 - 1$

(c) $e^3 - 4$ $\leftarrow$ correct

(d) $e^3 - 5$

(e) ∞

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^3 = \sum_{n=0}^{\infty} \frac{3^n}{n!} = 1 + 3 + \sum_{n=2}^{\infty} \frac{3^n}{n!}$

Common Exam III Prep. Part II

Find and simplify the Taylor series for $f(x) = \dfrac{1}{x^2}$ at $a = 3$.

$$f(x) = x^{-2}$$
$$f'(x) = -2x^{-3}$$
$$f''(x) = (-2)(-3)x^{-4}$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^n (n+1)!\, x^{-(n+2)}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)! 3^{-(n+2)}}{n!}(x-3)^n$$
$$= \sum_{n=0}^{\infty} (-1)^n (n+1) 3^{-(n+2)} (x-3)^n$$

Common Exam III Prep. Part II

(4 points) Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\cos(n^3)}{n^3}$$

$\sum_{n\geq 1} \left|\frac{\cos(n^3)}{n^3}\right| \leq \sum_{n\geq 1} \left|\frac{1}{n^3}\right|$ converges (p series)

By absolute convergence thm, $\sum_{n\geq 1} \frac{\cos(n^3)}{n^3}$ converges

Common Exam III Prep. Part II

(10 points) Determine whether the following series is absolutely convergent, conditionally convergent, or divergent. Show all work, as illustrated in class, by naming the test(s), applying the test(s), and drawing the correct conclusion(s).

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$$

$\sum_{n\geq 1} \frac{(-1)^n n}{n^2+1}$ converges (alternating series)

$\sum_{n\geq 1} \left|\frac{(-1)^n n}{n^2+1}\right|$ diverges (limit comparison)

$\sum_{n\geq 1} \frac{(-1)^n n}{n^2+1}$ converges conditionally

Common Exam III Prep. Part II

(10 points) Find the radius and interval of convergence of the series. Be sure to test the endpoints of the interval of convergence, and show all work by naming the test(s), applying the test(s), and drawing the correct conclusion(s).

$$\sum_{n=1}^{\infty} \frac{(-1)^n(2x+3)^n}{\sqrt{n}\,5^n}$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(2x+1)^{n+1}}{\sqrt{n+1}\cdot 5^{n+1}}\cdot\frac{\sqrt{n}\cdot 5^n}{(-1)^n(2x+1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2x+1)}{5}\cdot\frac{\sqrt{n}}{\sqrt{n+1}}\right| = \left|\frac{2x+1}{5}\right| < 1$$

$$\left|x+\frac{1}{2}\right| < \frac{5}{2} \leftarrow RoC$$

$$-\frac{5}{2} < x + \frac{1}{2} < \frac{5}{2}$$

$$-3 < x < 2$$

@ $x = -3$, $\sum_{n=0}^{\infty} \frac{(-1)^n(-5)^n}{\sqrt{n}5^n} =$, $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$:diverges

@ $x = 2$, $\sum_{n=0}^{\infty} \frac{(-1)^n(5)^n}{\sqrt{n}5^n} =$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$: converges

IoC : $(-3,2]$

Common Exam III Prep. Part II

(10 points) Suppose $f'(x) = \dfrac{1}{3+x^4}$. Find a MacLaurin Series for $f(x)$ if $f(0) = 5$. What is the radius of convergence of this series? Simplify your answer.

$$f' = \frac{1}{3} \cdot \frac{1}{1+\frac{x^4}{3}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x^4}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{3^{n+1}}$$

$$\int_0^x f'(x)dx = \sum_{n=0}^{\infty}\int_0^x (-1)^n \frac{x^{4n}}{3^{n+1}}dx$$

$$f(x) - 5 = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{4n+1}\frac{x^{4n+1}}{3^{n+1}}\right]_0^x$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{4n+1}\frac{x^{4n+1}}{3^{n+1}}$$

$$f(x) = 5 + \sum_{n=0}^{\infty}\frac{(-1)^n}{4n+1}\frac{x^{4n+1}}{3^{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{4n+5}\frac{x^{4n+5}}{3^{n+2}}\frac{(4n+1)(3^{n+1})}{x^{4n+1}}$$

$$= \lim_{n\to\infty}\frac{x^4}{3} = \frac{x^4}{3} < 1$$

$$x^4 < 3 \Rightarrow |x| < \sqrt[4]{3}$$