$0 < \theta < \frac{\pi}{2}$

1. Given the vectors $\mathbf{a} = \langle 1, -3 \rangle$ and $\mathbf{b} = \langle -3, 4 \rangle$. Find
   (a) The scalar and vector projections of $\mathbf{a}$ onto $\mathbf{b}$
   (b) The scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$

(a) $\vec{a}\cdot\vec{b} = \langle 1, -3 \rangle \cdot \langle -3, 4 \rangle = 1(-3) - 3(4) = -15$

$|\vec{a}| = \sqrt{1+9} = \sqrt{10}$

$|\vec{b}| = \sqrt{9+16} = \sqrt{25} = 5$

$\text{comp}_{\vec{b}}\,\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{-15}{5} = \boxed{-3}$ scalar projection

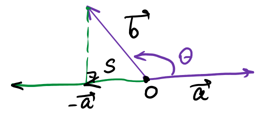
$\text{proj}_{\vec{b}}\,\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\,\vec{b} = \dfrac{-15}{25}\langle -3, 4 \rangle = -\dfrac{3}{5}\langle -3, 4 \rangle$

$= \boxed{\left\langle \dfrac{9}{5}, -\dfrac{12}{5} \right\rangle}$ vector projection

(b) $\text{comp}_{\vec{a}}\,\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \boxed{-\dfrac{15}{\sqrt{10}}}$ scalar projection

$\text{proj}_{\vec{a}}\,\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a} = -\dfrac{15}{10}\langle 1, -3 \rangle = -\dfrac{3}{2}\langle 1, -3 \rangle$

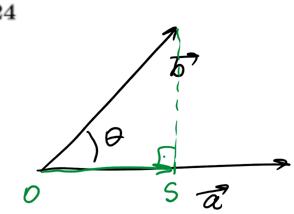
$= \boxed{\left\langle -\dfrac{3}{2}, \dfrac{9}{2} \right\rangle}$ vector projection



$\vec{OS} = \text{proj}_{\vec{a}}\,\vec{b}$

$\vec{OS}$ and $\vec{a}$ have opposite directions.

$\text{comp}_{\vec{a}}\,\vec{b} = -|\vec{OS}|$



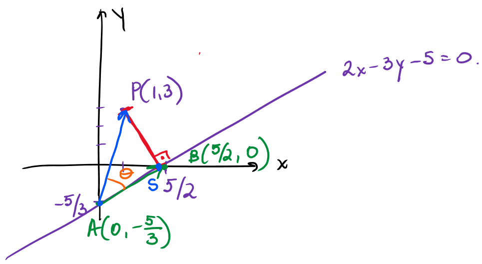
$\text{proj}_{\vec{a}}\,\vec{b} = \vec{OS}$ is the vector projection of $\vec{b}$ onto $\vec{a}$

$|\vec{OS}| = \text{comp}_{\vec{a}}\,\vec{b}$

$\boxed{\text{comp}_{\vec{a}}\,\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$ scalar projection

$\boxed{\text{proj}_{\vec{a}}\,\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}}$ vector projection.

2. Find the distance from the point $P(1,3)$ to the line $2x-3y-5=0$.



$\overrightarrow{PS}$ is perpendicular to $\overrightarrow{AB}$

distance $= \left|comp_{\overrightarrow{PS}} \overrightarrow{AP}\right|$

$\overrightarrow{PS}$ is the orthogonal complement to $\overrightarrow{AB}$

$\overrightarrow{PS} = \overrightarrow{AB}^{\perp}$

$A(0,-5/3),\ B(\frac{5}{2},0) \Rightarrow \overrightarrow{AB} = \underbrace{< \frac{5}{2}-0,\ 0-(-\frac{5}{3}) >}^{[end]-[start]}$

$\overrightarrow{AB} = < \frac{5}{2},\ \frac{5}{3} >$

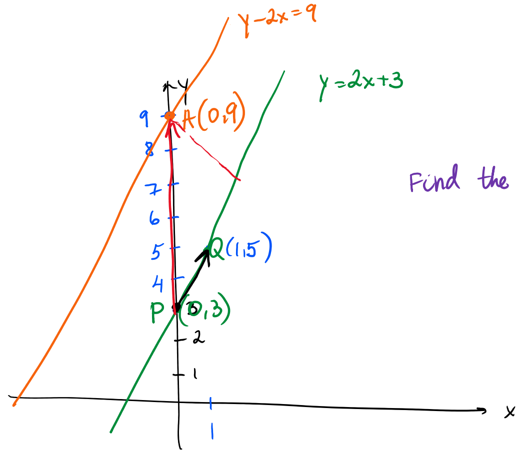
$\overrightarrow{PS} = \overrightarrow{AB}^{\perp} = \boxed{< \frac{5}{3}, -\frac{5}{2} >}$

$A(0,-5/3),\ P(1,3)$ $\quad \overrightarrow{AP} = < 1-0,\ 3-(-\frac{5}{3}) > = \boxed{< 1, \frac{14}{3} >}$

$$\text{dist} = \left|comp_{\overrightarrow{AB}^{\perp}} \overrightarrow{AP}\right| = \left|\frac{\overrightarrow{AB}^{\perp} \cdot \overrightarrow{AP}}{|\overrightarrow{AB}^{\perp}|}\right| = \left|\frac{< \frac{5}{3}, -\frac{5}{2} > \cdot < 1, \frac{14}{3} >}{\sqrt{\frac{25}{9}+\frac{25}{4}}}\right|$$

$$= \left|\frac{\frac{5}{3}-\frac{5}{2}\cdot\frac{14}{3}}{\sqrt{\frac{4\cdot 25+9\cdot 25}{36}}}\right| = \left|\frac{\frac{5}{3}-\frac{35}{3}}{\sqrt{\frac{13\cdot 25}{36}}}\right| = \left|\frac{-10}{\frac{5}{6}\sqrt{13}}\right| = \left|\frac{-10\cdot 6}{5\sqrt{13}}\right| = \left|-\frac{12}{\sqrt{13}}\right| = \boxed{\frac{12}{\sqrt{13}}}$$

3. Find the distance between the parallel lines $y = 2x + 3$ and $y - 2x = 9$.



distance between these lines = distance from a point on the line $y-2x=9$ to the line $y=2x+3$.

Find the dist from $A(0,9)$ to $y=2x+3$.

$$d = \left| \text{comp}_{\vec{PQ}^{\perp}} \vec{PA} \right|$$

$P(0,3)$, $Q(1,5)$ $\Rightarrow$ $\vec{PQ} = \langle 1-0, 5-3 \rangle$

$\vec{PQ} = \langle 1, 2 \rangle$

$\boxed{\vec{PQ}^{\perp} = \langle 2, -1 \rangle}$

$P(0,3)$, $A(0,9)$

$\boxed{\vec{PA} = \langle 0, 9-3 \rangle = \langle 0, 6 \rangle}$

$$d = \left| \text{comp}_{\vec{PQ}^{\perp}} \vec{PA} \right| = \left| \frac{\vec{PA} \cdot \vec{PQ}^{\perp}}{|\vec{PQ}^{\perp}|} \right| = \left| \frac{\langle 2,-1 \rangle \cdot \langle 0,6 \rangle}{|\langle 2,-1 \rangle|} \right| = \left| \frac{0-6}{\sqrt{4+1}} \right| = \left| \frac{-6}{\sqrt{5}} \right| = \boxed{\frac{6}{\sqrt{5}}}$$

4. Find a Cartesian equation for the following parametric curves. Sketch the curve.

(a) $x = 1 - t^2$, $y = 1 - t$, $-1 \le t \le 1$

(b) $x = 1 + \sin t$, $y = 2 + \cos t$

(c) $x = \tan t$, $y = \cot^2 t$, $\frac{\pi}{6} \le t \le \frac{\pi}{3}$

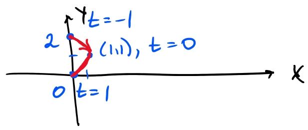
(a) $x = 1 - t^2$, $y = 1 - t$, $\boxed{-1 \le t \le 1}$

Eliminate t.

$y = 1 - t \Rightarrow t = 1 - y$, plug into $x = 1 - t^2$: $x = 1 - (1-y)^2$

$x = \cancel{1} - (\cancel{1} - 2y + y^2)$

$\boxed{x = 2y - y^2}$ parabola



$x = 1 - t^2$
$y = 1 - t$

$t = -1$	$x(-1) = 1 - 1 = 0$ $y(-1) = 1 - (-1) = 2$
$t = 0$	$x(0) = 1$ $y(0) = 1$
$t = 1$	$x(1) = 1 - 1 = 0$ $y(1) = 1 - 1 = 0$

(b) $x = 1 + \sin t$, $y = 2 + \cos t$ $\quad 0 \le t \le 2\pi$

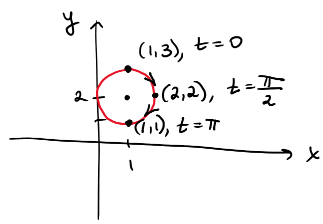
$\sin^2 t + \cos^2 t = 1$

$x = 1 + \sin t \Rightarrow \sin t = x - 1$

$y = 2 + \cos t \Rightarrow \cos t = y - 2$

$(x-1)^2 + (y-2)^2 = 1$

circle centered @ $(1,2)$ of radius 1



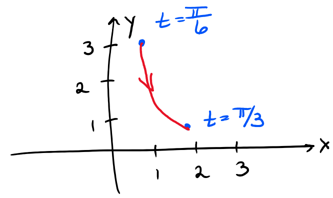
$t = 0$	$x(0) = 1 + \cancel{\sin 0}^{0} = 1$, $y(0) = 2 + \cancel{\cos 0}^{1} = 3$
$t = \frac{\pi}{2}$	$x\left(\frac{\pi}{2}\right) = 1 + \cancel{\sin \frac{\pi}{2}}^{1} = 2$, $y\left(\frac{\pi}{2}\right) = 2 + \cancel{\cos \frac{\pi}{2}}^{0} = 2$
$t = \pi$	$x(\pi) = 1 + \cancel{\sin \pi}^{0} = 1$, $y(\pi) = 2 + \cancel{\cos \pi}^{-1} = 1$

(c) $x = \tan t$, $y = \cot^2 t$, $\frac{\pi}{6} \le t \le \frac{\pi}{3}$

$\cot t = \frac{1}{\tan t}$ or $\tan t = \frac{1}{\cot t}$

$y = \cot^2 t = \frac{1}{\tan^2 t} = \frac{1}{x^2}$

$\boxed{y = \frac{1}{x^2}}$



$t = \frac{\pi}{6}$	$x\left(\frac{\pi}{6}\right) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$	$y\left(\frac{\pi}{6}\right) = \cot^2 \frac{\pi}{6} = (\sqrt{3})^2 = 3$
$t = \frac{\pi}{3}$	$x\left(\frac{\pi}{3}\right) = \tan \frac{\pi}{3} = \sqrt{3}$	$y\left(\frac{\pi}{3}\right) = \cot^2 \frac{\pi}{3} = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$

5. An object is moving in the xy-plane and its position after t seconds is $\mathbf{r}(t) =< t^2 + t, t - 4 >$.

(a) At what time is the object at the point (12,-1).

(b) Does the object pass through the point (4,8)?

(c) Find an equation in x and y whose graph is the path of the object.

(a) Find t such that $\vec{r}(t) = <12, -1>$

or $< t^2+t, t-4> = <12, -1>$

match up the components: $\begin{cases} t^2+t=12 \\ t-4=-1 \end{cases} \Rightarrow t=-1+4=3$

plug $t=3$ into the first equation: $t^2+t=12$

$\boxed{L.H.S.}$ $3^2+3=9+3=12$

$\boxed{L.H.S. = R.H.S.}$

$\boxed{\vec{r}(3)=<12,-1>}$ $\boxed{t=3}$

(b) Find t such that $\vec{r}(t)=<4,8>$

$<t^2+t, t-4> = <4, 8>$

$\begin{cases} t^2+t=4 \\ t-4=8 \end{cases} \Rightarrow t=8+4=12$

plug $t=12$ into the first equation $t^2+t=4$

$\boxed{L.H.S.}$ $12^2+12 = 144+12=156 \neq 4$

The object $\boxed{\text{DOES NOT}}$ pass through (4,8)

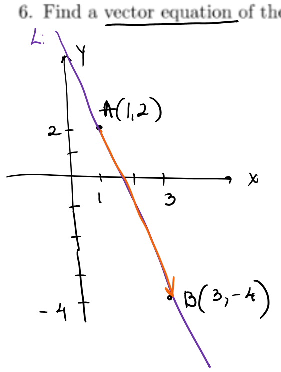
(c) $\vec{r}(t)=< t^2+t, t-4> \Rightarrow \begin{cases} x=t^2+t \\ y=t-4 \end{cases} \rightarrow t=y+4$ (plug into the equation for x)

$\boxed{x=(y+4)^2+(y+4)}$

$x = y^2+8y+16+y+4$

$\boxed{x=y^2+9y+20}$ parabola.

6. Find a vector equation of the line containing the points A(1,2) and B(3,-4).



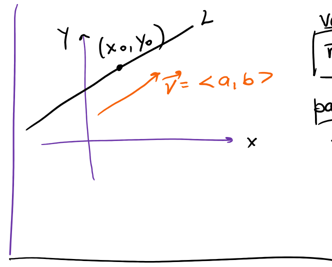
missing a vector parallel to the line.

the vector $\overrightarrow{AB}$ is parallel to the line

$\overrightarrow{AB} = \langle 3-1, -4-2 \rangle$

$\overrightarrow{AB} = \langle 2, -6 \rangle$

$\boxed{\vec{r}(t) = \langle 1,2 \rangle + t\langle 2,-6 \rangle}$ or $\boxed{\vec{r}(t) = \langle 3,-4 \rangle + t\langle 2,-6 \rangle}$



vector equation

$\boxed{\vec{r}(t) = \langle x_0, y_0 \rangle + t\langle a,b \rangle}$

parametric equations:

$\langle x, y \rangle = \langle x_0, y_0 \rangle + t\langle a, b \rangle$

$$\boxed{\begin{cases} x = x_0 + at \\ y = y_0 + bt \end{cases}}$$

7. Find parametric equations of the line passing through the point (-1,1) and parallel to the vector $\vec{\imath} - 5\vec{\jmath}$.

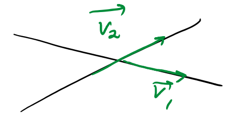
$\vec{v} = \langle 1, -5 \rangle$

$\langle x, y \rangle = \vec{r}(t) = \langle -1, 1 \rangle + t \langle 1, -5 \rangle$

$\langle x, y \rangle = \langle -1 + t, 1 - 5t \rangle$

match up the corresponding components: $\boxed{\begin{cases} x = -1 + t \\ y = 1 - 5t \end{cases}}$ parametric equations.

8. Determine whether the lines $\mathbf{r}(t) = (-4+2t)\vec{\imath} + (5+t)\vec{\jmath}$ and $\mathbf{r}(t) = (2+3t)\vec{\imath} + (4-6t)\vec{\jmath}$ are parallel, perpendicular or neither. If they are not parallel, find their point of intersection.



L_1: $\vec{r}(t) = \langle -4+2t,\ 5+t \rangle$
$= \langle -4, 5 \rangle + \langle 2t, t \rangle$
$= \langle -4, 5 \rangle + t\langle 2, 1 \rangle$

Passes through $(-4,5)$ and is parallel to the vector $\vec{v}_1 = \langle 2, 1 \rangle$

L_2: $\vec{r}(t) = \langle 2+3t,\ 4-6t \rangle$
$= \langle 2, 4 \rangle + \langle 3t, -6t \rangle$
$= \langle 2, 4 \rangle + t\langle 3, -6 \rangle$

Passes through $(2,4)$ and is parallel to the vector $\vec{v}_2 = \langle 3, -6 \rangle$

$\vec{v}_1 = \langle 2, 1 \rangle$, $\vec{v}_2 = \langle 3, -6 \rangle$ not parallel

$\vec{v}_1 \cdot \vec{v}_2 = \langle 2, 1 \rangle \cdot \langle 3, -6 \rangle = 2(3) + 1(-6) = 6 - 6 = 0 \Rightarrow \vec{v}_1$ is perpendicular to $\vec{v}_2$

the lines are perpendicular as well

Point of intersection:

L_1: $\vec{r}_1(t) = \langle -4+2t,\ 5+t \rangle$, L_2: $\vec{r}_2(s) = \langle 2+3s,\ 4-6s \rangle$

In the point of intersection $\vec{r}_1(t) = \vec{r}_2(s)$

$$\langle -4+2t,\ 5+t \rangle = \langle 2+3s,\ 4-6s \rangle$$

matching up the components:
$$\begin{cases} -4+2t = 2+3s \\ 5+t = 4-6s \end{cases}$$
solve for s and t

$t = 4-5-6s$

$t = -1-6s$, plug into the first equation.

$$-4+2\overbrace{(-1-6s)}^{t} = 2+3s$$
$$-6-12s = 2+3s \Rightarrow -8 = +15s \Rightarrow \boxed{s = -\frac{8}{15}}$$
$$t = -1-6\left(-\frac{8}{15}\right) = -1 + \frac{48}{15} = \frac{33}{15}$$

$\boxed{\text{Point of intersection}}$

$$\vec{r}_2\left(s = -\frac{8}{15}\right) = \left\langle 2 + 3\left(-\frac{8}{15}\right),\ 4 - 6\left(-\frac{8}{15}\right) \right\rangle$$
$$= \left\langle 2 - \frac{24}{15},\ 4 + \frac{48}{15} \right\rangle$$
$$= \left\langle \frac{6}{15},\ \frac{108}{15} \right\rangle$$

Point of intersection $= \boxed{\left\langle \frac{2}{5},\ \frac{36}{5} \right\rangle}$

9. Find the exact value of the expression.

a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

b) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

c) $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

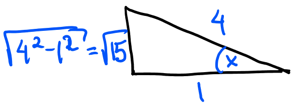
d) $\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

e) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

f) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

g) $\sin\underbrace{\left(\arccos\frac{1}{4}\right)}_{x} = \sin x$

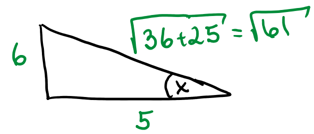
$\arccos\frac{1}{4} = x \Rightarrow \cos x = \frac{1}{4}$



$\boxed{\sin x = \frac{\sqrt{15}}{4}}$

h) $\cos\underbrace{\left(\arctan\frac{6}{5}\right)}_{x} = \cos x$

$\arctan\frac{6}{5} = x$ or $\tan x = \frac{6}{5}$



$\boxed{\cos x = \frac{5}{\sqrt{61}}}$

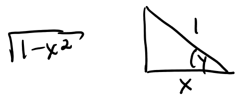
10. Simplify the expression

(a) $\tan\left(\cos^{-1} x\right)$

(b) $\sin\left(\tan^{-1} x\right)$

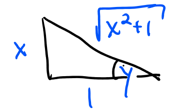
(a) $\tan(\underbrace{\cos^{-1} x}_{y}) = \tan y = \boxed{\dfrac{\sqrt{1-x^2}}{x}}$

$\cos^{-1} x = y \Rightarrow \cos y = x$



(b) $\sin(\underbrace{\tan^{-1} x}_{y}) = \sin y,$ when $y = \tan^{-1} x$

$\tan y = x$



$= \boxed{\dfrac{x}{\sqrt{x^2+1}}}$