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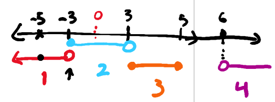
Section 5.6: Exponential Functions

- Exponential Function
- Exponential Growth and Exponential Decay
- Common Base Property of Exponents: For $b \neq 1$, $b^S = b^T$ if and only if $S = T$.
- Finance Applications

Piecewise- defined

Pr 1. Let $f(x) = \begin{cases} 3x-1 & x < -3 \\ x^2-1 & -3 \le x < 3 \\ 7 & 3 \le x \le 5 \\ \frac{1}{x-5} & x > 6 \end{cases}$. Compute the following function values.



(a) $f(-5)$.

-5 is in piece 1

$f(-5) = 3(-5) - 1 = -15-1 = \boxed{-16}$

(b) $f(-3)$.

-3 is in piece 2

$f(-3) = (-3)^2 - 1 = 9-1 = \boxed{8}$

$-3 < -3$? No

$-3 \le -3 < 3$

wrong answer

$3(-3)-1 = -9-1 = \boxed{-10}$

(c) $f(0)$. 0 is in piece 2

$f(0) = (0)^2 - 1 = 0-1 = \boxed{-1}$

(d) $f(3)$. 3 is on piece 3

$f(3) = \boxed{7}$

$y=7$ is a horizontal line

$f(x)=7$, then $f(3)=7$.

(e) $f(6)$. 6 is not on any piece

f(6) DNE or is undefined.

$6 > 6$ is false

f(5.5) is also undefined.

(f) $f(7)$.

7 is on piece 4 (rule $\frac{1}{x-5}$)

$f(7) = \frac{1}{(7)-5} = \boxed{\frac{1}{2}}$

1) find domain of each piece
2) Take the union

Pr 2. State the domain of $g(x)$.

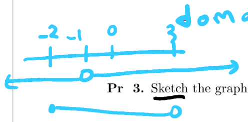
$$g(x) = \begin{cases} x+1 & x < -3 \\ \frac{1}{x+1} & -2 \le x < 3 \\ \sqrt{x-2} & x \ge 3 \end{cases}$$

$(-\infty, -3)$
$[-2, 3)$
$[3, \infty)$

domain $(x+1) \cap (-\infty,-3] = (-\infty,\infty) \cap (-\infty,-3)$
$= (-\infty,-3)$

domain $\left(\frac{1}{x+1}\right) \cap [-2,3)$ $\quad x+1 \neq 0 \to x \neq -1$

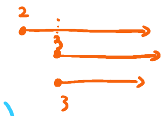
$((-\infty,-1) \cup (-1,\infty)) \cap [-2,3) = [-2,-1) \cup (-1,3)$



domain $(\sqrt{x-2}) \cap [3,\infty)$

$x-2 \ge 0 \to x \ge 2$

$[2,\infty) \cap [3,\infty) =$
$[3,\infty)$



$(-\infty,-3) \cup [-2,-1) \cup (-1,3) \cup [3,\infty)$

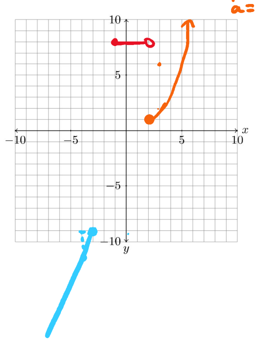
$= \boxed{(-\infty,-3) \cup [-2,-1) \cup (-1,\infty)}$

Pr 3. Sketch the graph of $h(x)$.

$$h(x) = \begin{cases} 2x-3 & x \le -3 \\ 8 & -1 \le x < 2 \\ x^2-3 & x \ge 2 \end{cases}$$

$a = 1 > 0$

remark:
if $x < -3$ $\rightarrow$ hole at $(-3,-9)$



$2x-3$
slope is 2
$h(-3) = 2\cdot(-3)-3 = -6-3 = -9$

$h(2) = 2^2-3 = 4-3 = 1$
$(2,1)$

$y = x^2$

$h(3) = 3^2-3 = 9-3 = 6$

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases} \quad \text{or} \quad |x| = \begin{cases} -x & x \leq 0 \\ x & > 0 \end{cases}$$

Pr 4. Rewrite $f(x) = |5 - 3x|$ as a piecewise-defined function.

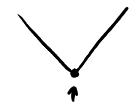
$$|5-3x| = \begin{cases} -(5-3x) & \text{if } 5-3x < 0 \\ 5-3x & \text{if } 5-3x \geq 0 \end{cases}$$

$5-3x < 0$

$-5 \quad -5$

$\frac{-3x}{-3} < \frac{-5}{-3} \rightarrow x > \frac{5}{3}$

$x \leq \frac{5}{3}$



$$|5-3x| = \begin{cases} 5-3x & x \leq \frac{5}{3} \\ -5+3x & x > \frac{5}{3} \end{cases} \quad \text{or} \quad |5-3x| = \begin{cases} 5-3x & x < \frac{5}{3} \\ -5+3x & x \geq \frac{5}{3} \end{cases}$$

Pr 5. Suppose that on the island of St. Thomas, the cost of electricity is \$0.12 per kilowatt for the first 3000 kilowatts a household uses, per month. After 3000 kilowatts, the cost increases to \$0.20 per additional kilowatt used during the month. Write the function, $C(e)$, representing the cost, C, for e kilowatts of electricity used by a household in this country in a month.

$$C(e) = \begin{cases} .12e & \text{if } e \leq 3000 \\ .2e - 600 & \text{if } e > 3000 \end{cases}$$

wrong answer: $C(e) = .2e \rightarrow$ overcharging

3001 kw

charge $= .12 \times 3000 + .20$

$$C(e) = \underbrace{.12 \times 3000}_{\text{charge for 1st 3000}} + .2\,(\underbrace{e - 3000}_{\text{amount over 3000}})$$

$= .12 \times 3000 + .2e - .2 \times 3000$

$= 360 + .2e - 600 = .2e - 600$

power functions $f(x) = x^p$

exponential $f(x) = b^x$

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Pr 1. Rewrite each exponential expression as a single equivalent expression in the stated base.

answer $= 5^{\text{somethin}}$

$a^b \cdot a^c = a^{b+c}$

(a) $125 \cdot 5^{x+3}$, base 5.

$125 \cdot 5^{x+3} = 5^3 \cdot 5^{x+3} = 5^{3+x+3} = \boxed{5^{x+6}}$

$5^1, 5^2, 5^3 \ldots$

$5 \to 25 \to 125$

(b) $\left(\frac{1}{2}\right)^x \frac{8}{4^x}$, base 2.

$2^{\sim}$

$\left(\frac{1}{2}\right)^x \frac{8}{4^x} = 2^{-x} \cdot \frac{8}{4^x} = 2^{-x} \cdot 8 \cdot 4^{-x}$

$= 2^{-x} \cdot 2^3 \cdot (2^2)^{-x}$

$\left(\frac{1}{a}\right)^b = (a^{-1})^b = a^{-b}$

$\frac{1}{4^x} = 4^{-x}$

$2^a \cdot 4^b \neq 2^{a+b}$

$4 = 2^2$

$8 = 2^3$

$(a^b)^c = a^{b\cdot c}$

$2^{-x+3} \cdot 2^{2\cdot(-x)}$

$= 2^{-x+3} \cdot 2^{-2x}$

$= 2^{-x+3-2x}$

$= \boxed{2^{-3x+3}}$

Approach II:

$\left(\frac{1}{2}\right)^x \cdot 8 \cdot \frac{1}{4^x}$

$(2^{-1})^x \cdot 2^3 \cdot \frac{1}{(2^2)^x} = 2^{-x} \cdot 2^3 \cdot (2^2)^{-x}$

$= 2^{-x} \cdot 2^3 \cdot 2^{-2x}$

$= 2^{-x+3-2x}$

$= \boxed{2^{-3x+3}}$

Pr 2. Determine if each function is an exponential function. If the function is an exponential function, determine whether the function represents exponential growth or decay.

a^x growth if $a > 1$

decay if $0 < a < 1$.

(a) 7^{-x}

$= (7^{-1})^x = \left(\frac{1}{7}\right)^x$

or $\frac{1}{7^x} = \left(\frac{1}{7}\right)^x$ is exponential ✓

exponential decay

(b) $-3x^{17}$

$x^{17} = \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdots x}_{17 \text{ times}}$ is a power function

not exponential.

x^x is not exponential.

$y = ae^{kx}$

$k > 0$ growth

$k < 0$ decay

(c) $\frac{3}{4} 2^{x/2+4}$

$= \frac{3}{4} 2^{x/2} \cdot 2^4 = 2^4 \cdot \frac{3}{4}\, 2^{x/2}$

a^x

$x/2 = x \cdot \frac{1}{2}$

$= \frac{1}{2} \cdot x$

$a^{b+c} = a^b \cdot a^c$

$= 2^4 \cdot \frac{3}{4} \left(2^{1/2}\right)^x$

$= 12 \left(\sqrt{2}\right)^x$

$y = a \cdot b^x$

this is exponential growth

$\sqrt{2} \overset{?}{>} 1$ ✓

Pr 3. State the domain, range, end behavior, x-intercepts, and y-intercepts of each function.

(a) $f(x) = \left(\frac{5}{3}\right)^{x+2}$

$f(x) = 1\cdot\left(\frac{5}{3}\right)^{x+2}$ $\quad 5/3 > 1$ $\quad$ growth. $\quad a > 0$

domain: $(-\infty, \infty)$

range: $(0, \infty)$

x-intercepts: DNE

y-intercept: $f(0) = \left(\frac{5}{3}\right)^{0+2} = \left(\frac{5}{3}\right)^2 = \frac{5^2}{3^2} = \frac{25}{9}$

$= \frac{5}{3}\cdot\frac{5}{3} = \frac{5\cdot 5}{3\cdot 3}$

$\boxed{\left(0, \frac{25}{9}\right)}$

end behavior: as $x \to \infty$, $f(x) \to \infty$

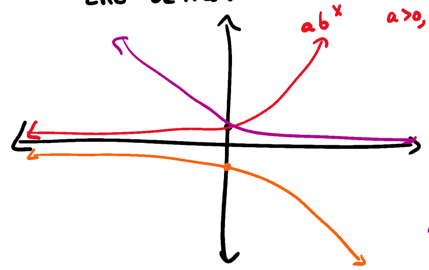
as $x \to -\infty$, $f(x) \to 0$.

$y = a^x \quad a > 1 \qquad y = -c\cdot a^x$

domain: $(-\infty, \infty)$

range: $(0, \infty) \qquad (-\infty, 0)$

End-behavior



$a>0,\ b>1$

as $x \to -\infty$, $f(x) \to 0$

as $x \to \infty$, $f(x) \to \infty$

$a>0\ \ 0<b<1$

flip around y-axis

as $x \to \infty$, $f(x) \to 0$

$ab^x \quad a<0,\ b>1$

(b) $g(x) = 5^{4-x}$ $= 5^4 \cdot 5^{-x} = 5^4 \cdot \left(\frac{1}{5}\right)^x$ $a > 0$ $\frac{1}{5} < 1$, decay

domain: $(-\infty, \infty)$
range: $(0, \infty)$
end behavior: as $x \to \infty$, $g(x) \to 0$.
as $x \to -\infty$, $g(x) \to \infty$

x-intercept: DNE.
y-intercept: $g(0) = 5^{4-0} = 5^4 = 625$
$(0, 625)$

Pr 4. State the domain of each function. Write your answer using interval notation.

(a) $f(x) = 5^{\frac{2x}{x-4}}$

domain $\left(5^{\frac{2x}{x-4}}\right)$ = domain $\left(\frac{2x}{x-4}\right)$

$\boxed{(-\infty, 4) \cup (4, \infty)}$

domain of $a^{f(x)}$ is the domain of $f(x)$.

$x - 4 \neq 0$

$x - 4 = 0$

$x = 4$

$x \neq 4$

refresher : $\frac{f(x)}{g(x)}$

need $g(x) \neq 0$

(b) $g(x) = e^{\sqrt{1-4x}}$

domain $\left(e^{\sqrt{1-4x}}\right)$ = domain $\left(\sqrt{1-4x}\right)$

e is a constant

need $1 - 4x \geq 0$

$1 \geq 4x$

$\frac{1}{4} \geq x$ or $x \leq \frac{1}{4}$

$\boxed{\left(-\infty, \frac{1}{4}\right]}$

$\left[\frac{1}{4}, \infty\right)$ is a common wrong choice.

$\sqrt{f(x)}$ need $f(x) \geq 0$

$\sqrt[4]{f(x)}$

$\vdots$

$\sqrt[2k]{f(x)}$

(c) $h(a) = \frac{\sqrt[3]{2x-5}}{3^{x+2}}$ = ratio = $\sqrt[3]{2x-5} \cdot 3^{-(x+2)}$

$= \sqrt[3]{2x-5} \left(\frac{1}{3}\right)^{x+2}$ decay

exponential

worried about denominator

$3^{x+2} = 0$? ← no solution

domain $\left(\sqrt[3]{2x-5}\right)$

= domain $(2x-5)$ = $(-\infty, \infty)$

domain $\left(\left(\frac{1}{3}\right)^{x+2}\right)$ = domain $(x+2)$

$= (-\infty, \infty)$

first guess: $\boxed{(-\infty, \infty)}$

domain $(f(x) \cdot g(x))$

= domain $f(x)$ $\cap$ domain $g(x)$

not using graphing calc.

Pr 5. Algebraically solve each equation for x.

(a) $\left(\frac{1}{4}\right)^{2x} = 8^{x-5}$.

strategy : rewrite as $a^y = a^z$

$a^S = a^T$ iff $S = T$.

$$\left(\frac{1}{4}\right)^{2x} = 8^{x-5}$$

possible bases: $4, 8, 2$ ($4 = 2^2$, $8 = 2^3$)

$$\left(\frac{1}{2^2}\right)^{2x} = (2^3)^{(x-5)} \rightarrow 2^{3\cdot(x-5)} = 2^{3x-15}$$

common mistake

$(2^3)^{x+5} = 2^{3x+5}$

$$\left(\frac{1}{2^2}\right)^{2x} = (2^{-2})^{2x} = 2^{-4x}$$

$$2^{-4x} = 2^{3x-15} \longrightarrow -4x = 3x - 15$$

$$-3x \quad -3x$$

$$-7x = -15$$

$$\boxed{x = 15/7}$$

(b) $2^{2x}(2^x + 8) = \frac{2^x + 2^{3x}}{2^x}$

common base: 2, $8 = 2^3$

$$2^{2x}(2^x + 2^3) = \frac{2^x + 2^{3x}}{2^x} = \frac{2^x}{2^x} + \frac{2^{3x}}{2^x} = 1 + 2^{3x}\cdot 2^{-x}$$

$$= 1 + 2^{3x-x} = 1 + 2^{2x}$$

$$2^{2x}\cdot 2^x + 2^{2x}\cdot 2^3 = 2^{2x+x} + 2^{2x+3} = 2^{3x} + 2^{2x+3} \longrightarrow 2^{3x} + 2^{2x+3} = 1 + 2^{2x} \ ?$$

$2^a = 2^b$

$y = 2^x$

$$(2^x)^3 + 2^3\cdot(2^x)^2 = 1 + (2^x)^2$$

$$y^3 + 2^3 y^2 = 1 + y^2$$

$$y^3 + 8y^2 = 1 + y^2$$

$$y^3 + 7y^2 - 1 = 0.$$

solve for y ...

this is ugly...

usually you get

$y^2 + 4y + 4 = 0$

$(y+2)^2 = 0$ ...

(c) $(\frac{1}{25})^{3x} \cdot 5^{x^2} - 1 = 0$

common base: not 2

$25 = 5^2$, 5

$$\left(\frac{1}{5^2}\right)^{3x} \cdot 5^{x^2} = 1$$

$$(5^{-2})^{3x} \cdot 5^{x^2} = 5^{-6x} \cdot 5^{x^2}$$

$$= 5^{x^2 - 6x} = 1 \ (= 5^0)$$

$1 = a^0$

common mistake

$x^2 - 6x = 1$

$$x^2 - 6x = 0$$

$$x(x-6) = 0$$

$$\boxed{x = 0 \qquad x = 6}$$

Pr 6. If you invest \$2000 in an account that earns interest at a rate of 3.16% per year, compounded monthly, how much will be in the account after 10 years? If the annual interest is compounded continuously instead of monthly, how much more will be in the account after 10 years compared to your previous answer?

$3.16\% = .0316$

$$P(t) = P_0\left(1 + \frac{r}{12}\right)^{12t}$$

t is in years since initial investment

$$P(t) = P_0 e^{rt}$$

compounded continuously

monthly $\rightarrow$ $P(10) = 2000\left(1 + \frac{.0316}{12}\right)^{12\times 10}$

$$= 2000\left(1 + \frac{.0316}{12}\right)^{120} \approx \$2{,}742.12$$

cont. $\rightarrow$ $P(10) = 2000\, e^{.0316\cdot 10} = 2000\, e^{.316} \approx 2743.26$

$$\begin{array}{r} 2743.26 \\ -2742.12 \\ \hline \$1.14 \end{array}$$

Pr 7. If a company opens in 2018, and the company's revenue grows at an annual rate of 125% per year, the revenue function would be $R(t) = R_0(\frac{5}{4})^t$, where R_0 represents the initial revenue earned in 2018, and t represents the number of years since 2018. How much money did the company bring in, in revenue, in 2018, if the company's revenue is \$850,000 in 2023?

$= 1.25$

$$R(t) = R_0\left(\frac{5}{4}\right)^t$$

in 2023 $\quad R(5) = 850\,000$

$$\begin{array}{r} 2023 \\ -2018 \\ \hline 5 \end{array}$$

want $R(0)$

$$R(0) = R_0\left(\frac{5}{4}\right)^0 = R_0$$

$$850{,}000 = R(5) = R_0\left(\frac{5}{4}\right)^5 = R_0\frac{5^5}{4^5}$$

$$= R_0\frac{3125}{1024}$$

$$850{,}000 = R(5) = R_0\left(\frac{5}{4}\right)^5 = R_0\,\frac{5^5}{4^5}$$

$$= R_0\,\frac{3125}{1024}$$

$$R_0 = \frac{1024}{3125} \times 850{,}000 \quad \text{or} \quad 850000 \times 4^5/5^5$$

$$\boxed{R_0 = \$278{,}528}$$

next week: Exam review