Math 152 - Week-In-Review 11

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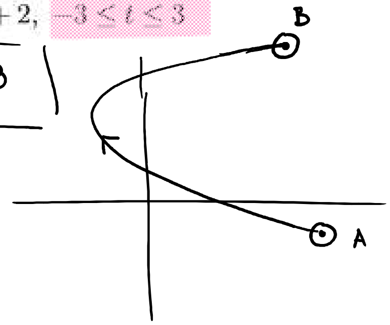
Eliminate the parameter to find the Cartesian equation of the curve.

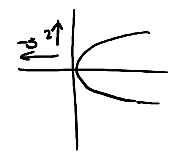
1. $x = t^2 - 3,\ y = t + 2,\ -3 \le t \le 3$

$\boxed{x = (y-2)^2 - 3}$



$x = t^2 - 3$
$y = t + 2$

$t = y - 2$



A @ $t = -3 \to (6, -1)$

B @ $t = 3 \to (6, 5)$

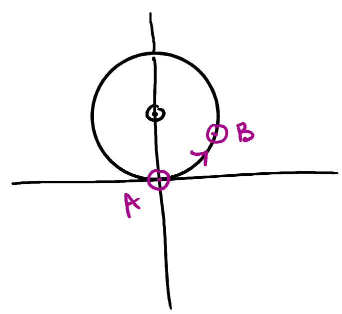
2. $x = \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi.$

Trig Identity
$\sin^2 t + \cos^2 t = 1$.

$\sin(t) = x \qquad \cos(t) = 1 - y$

$\boxed{x^2 + (1-y)^2 = 1}$

Circle with center @ $(0,1)$
radius $= 1$



A @ $t = 0 \to (0, 0)$

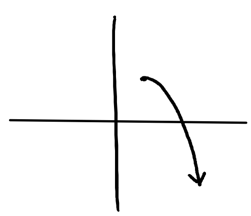
B @ $t = \pi/4 \to \left(\frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}\right)$

3. $x = \sqrt{t},\ y = 1 - t.$ (no bounds on t)

Substitution: $y = 1 - t$
$t = 1 - y$

$x = \sqrt{t}$

$\boxed{x = \sqrt{1-y}}$ unbounded



one complete circle

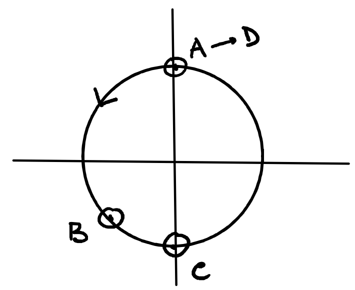
4. Sketch the curve given by $x = \sin 4\theta$, $y = \cos 4\theta$, $0 \leq \theta \leq \pi/2$ and indicate the direction of the curve that is traced as the parameter increases.

$x = \sin(4\theta)$ $\qquad$ $y = \cos(4\theta)$

$\frac{4\pi}{3} \to Q\text{III}$

$\sin^2(4\theta) + \cos^2(4\theta) = 1$.

$\boxed{x^2 + y^2 = 1}$



A @ $\theta = 0 \longrightarrow (0, 1)$

B @ $\theta = \pi/3 \longrightarrow (\sin \frac{4\pi}{3}, \cos \frac{4\pi}{3}) = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$

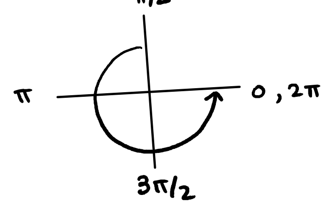
C @ $\theta = \pi/4 \longrightarrow (\sin \pi, \cos \pi) = (0, -1)$

D @ $\theta = \pi/2 \longrightarrow (\sin(2\pi), \cos(2\pi)) = (0, 1)$

going counterclockwise

5. Describe the motion of the particle with position (x, y) given as $x = 2 + \sin t$, ~~$y = 2 + 2\cos t$~~ $y = 1 + \cos t$, as t varies from $\pi/2$ to 2π.

$\frac{\pi}{2} \leq t \leq 2\pi$



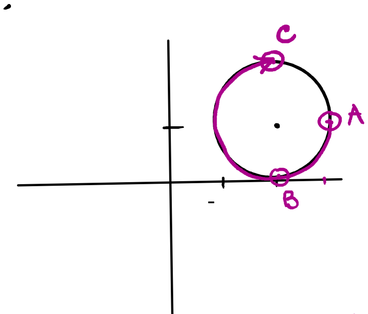
$x = 2 + \sin t$ $\qquad$ $y = 1 + \cos t$

$\sin t = x - 2$ $\qquad$ $\cos t = 1 - y$.

$\boxed{(x-2)^2 + (1-y)^2 = 1}$

circle with center @ $(2, 1)$

radius $= 1$.



A @ $t = \pi/2 \longrightarrow (3, 1)$

B @ $t = \pi \longrightarrow (2, 0)$

C @ $t = 2\pi \longrightarrow (2, 2)$

path is clockwise but not a complete circle.

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

6. Set up an integral to find the length of the part of the parametric curve given by $x = t + e^{-t}$, $y = t^2 + t$, $1 \le t \le 2$.

$x = t + e^{-t}$

$x' = 1 - e^{-t}$

$(x')^2 = 1 - 2e^{-t} + e^{-2t}$

$y = t^2 + t$

$y' = 2t + 1$.

$(y')^2 = 4t^2 + 4t + 1$

$$L = \int_1^2 \sqrt{1 - 2e^{-t} + e^{-2t} + 4t^2 + 4t + 1}\, dt$$

Evaluate the integral

7. Find the exact length of the curve $x = e^t - t$, $y = 4e^{t/2}$, $0 \le t \le 2$.

$x = e^t - t$

$x' = e^t - 1$

$(x')^2 = (e^t - 1)^2$

$= e^{2t} - 2e^t + 1$

$y = 4e^{t/2}$

$y' = 4e^{t/2} \cdot \frac{1}{2} = 2e^{t/2}$

$(y')^2 = (2e^{t/2})^2$

$= 4e^t$

$(x')^2 + (y')^2 = e^{2t} - 2e^t + 1 + 4e^t = e^{2t} + 2e^t + 1$

$= (e^t + 1)^2$

$$L = \int_0^2 \sqrt{(e^t+1)^2}\, dt = \int_0^2 (e^t + 1)\, dt = e^t + t \Big|_0^2$$

$= e^2 + 2 - (e^0 + 0)$

$\boxed{L = e^2 + 1}$

$S = \int_a^b 2\pi r \sqrt{(x')^2 + (y')^2}\, dt$ $\quad r = y(t)$ $\quad r = x(t)$

8. Set up an integral to represent the surface area obtained by rotating the curve $x = \sin t$, $y = \sin 2t$, $0 \le t \le \pi/2$ about the x-axis. $\rightarrow r = y(t) = \sin(2t)$

$x = \sin t \qquad y = \sin(2t)$

$x' = \cos t \qquad y' = 2\cos(2t)$

$(x')^2 = \cos^2 t \qquad (y')^2 = 4\cos^2(2t)$

$$S = 2\pi \int_0^{\pi/2} \sin(2t) \sqrt{\cos^2(t) + 4\cos^2(2t)}\, dt$$

9. Find the surface area obtained by rotating the curve $x = t^3$, $y = t^2$, $0 \le t \le 1$ about the x-axis. $\rightarrow r = y(t) = t^2$

$x = t^3 \qquad y = t^2$

$x' = 3t^2 \qquad y' = 2t$

$(x')^2 = 9t^4 \qquad (y')^2 = 4t^2$

$\sqrt{(x')^2 + (y')^2} = \sqrt{9t^4 + 4t^2} = \sqrt{t^2(9t^2+4)} = t\sqrt{9t^2+4}$

$$S = 2\pi \int_{t=0}^{t=1} t^2 \cdot t \underbrace{\sqrt{9t^2+4}}_{\sqrt{u}}\, dt$$

$\boxed{u = 9t^2 + 4}$ $\quad t=1,\ u=13$

$du = 18t\,dt \rightarrow t\,dt = \frac{du}{18}$ $\quad t=0,\ u=4$

$t^2 = \left(\frac{u-4}{9}\right)$

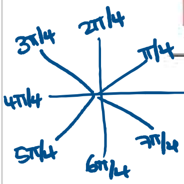
$$= 2\pi \int_{u=4}^{u=13} \left(\frac{u-4}{9}\right)\sqrt{u} \cdot \frac{du}{18}$$

$u\sqrt{u} = u^{3/2}$

$$= \frac{2\pi}{9(18)} \int_4^{13} (u\sqrt{u} - 4\sqrt{u})\, du = \frac{\pi}{81}\left[\frac{u^{5/2}}{5/2} - 4\frac{u^{3/2}}{3/2} \Bigg|_{u=4}^{u=13} \right]$$

$$= \frac{\pi}{81}\left[\frac{2}{5}(13^{5/2}) - 4\cdot\frac{2}{3}(13^{3/2}) - \frac{2}{5}(4^{5/2}) + 4\cdot\frac{2}{3}(4^{3/2}) \right]$$

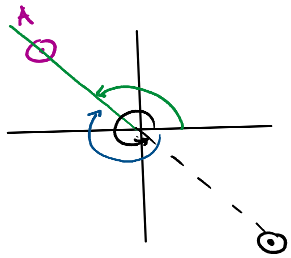
(x,y) $\quad x = r\cos\theta$, $y = r\sin\theta$ $\quad|\quad$ $x^2+y^2=r^2$, $\tan\theta = \frac{y}{x}$ $\quad (r,\theta)$



$\tan\theta = \left(\frac{4}{-4}\right) = (-1)$ $\quad \theta = 3\pi/4$ $\quad$ A $\quad$ a)

10. Give the polar coordinates for the cartesian point $(x, y) = (-4, 4)$ when $r > 0$ and when $r < 0$. b)

QII



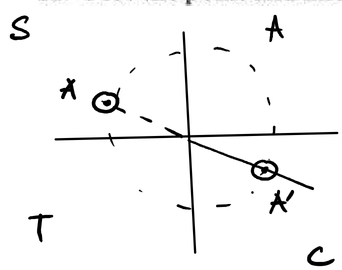
$r = \sqrt{x^2+y^2} = \sqrt{(-4)^2+(4)^2} = \sqrt{32} = \sqrt{2\cdot 16} = 4\sqrt{2}$

a) $r>0$ $\quad (r,\theta) = \left(4\sqrt{2}, \frac{3\pi}{4}\right) = \left(4\sqrt{2}, -\frac{5\pi}{4}\right)$

b) $r<0$ $\quad (r,\theta) = \left(-4\sqrt{2}, -\pi/4\right) = \left(-4\sqrt{2}, \frac{7\pi}{4}\right)$

A → QII

11. Plot the point and find the cartesian coordinates for the polar point $(r, \theta) = (-1, -\pi/6)$.



$r = -1$

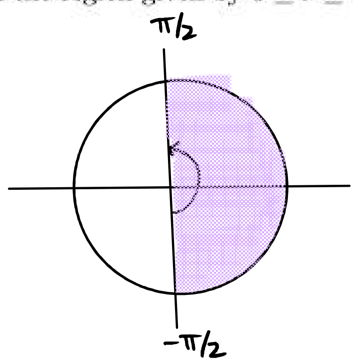
$x = r\cos\theta = (-1)\cos(-\pi/6)$
$= (-1)\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$

$y = r\sin\theta = (-1)\sin(-\pi/6) = +\frac{1}{2}$

$(r,\theta) \longrightarrow (x,y) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \to$ QII

12. Sketch the region given by $0 \le r \le 1$, $-\pi/2 \le \theta \le \pi/2$.

want to maintain continuity for integration



→ helps to identify bounds of integration

QI → $0 \le \theta \le \pi/2$

QII → $\pi/2 \le \theta \le \pi$

QIII → $\pi \le \theta \le 3\pi/2$

QIV → $3\pi/2 \le \theta \le 2\pi$ or $-\pi/2 \le \theta \le 0$

13. Find the cartesian equation for the polar curve $r^2 = 10$. $\longrightarrow$ $r = \sqrt{10}$

$r^2 = 10$

$\boxed{x^2 + y^2 = 10}$

circle with center @ $(0,0)$

radius $= \sqrt{10}$

14. Find the cartesian equation for the polar curve $r^2 \sin 2\theta = 1$.

$\sin(2\theta) = 2\sin\theta\cos\theta$

$r^2 \sin(2\theta) = 1$

$r \cdot r \cdot 2\sin\theta\cos\theta = 1$

$(r\sin\theta)(r\cdot\cos\theta) = \frac{1}{2}$

$\boxed{y \cdot x = \frac{1}{2}}$

$x = r\cos\theta$
$y = r\sin\theta$

15. Find the polar equation for the cartesian curve $x^2 + y^2 = 4y$.

$x^2 + (y^2 - 4y) = 0$

$x^2 + (y^2 - 4y + 4) = 4$

$x^2 + (y-2)^2 = 4$

circle of radius 2
center @ $(0,2)$

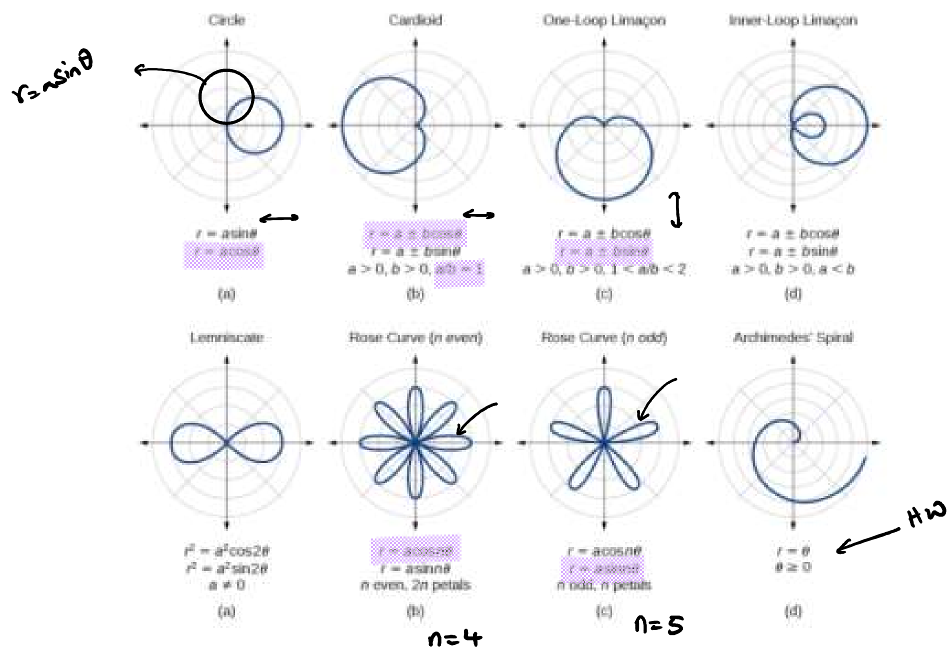
$x^2 + y^2 = 4y$

$r^2 = 4.(r\sin\theta)$

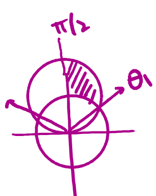
$\therefore r = 4\sin\theta$ $\sim a\sin\theta$

radius $= 2$ $= a/2$

Brief overview of polar Curves:

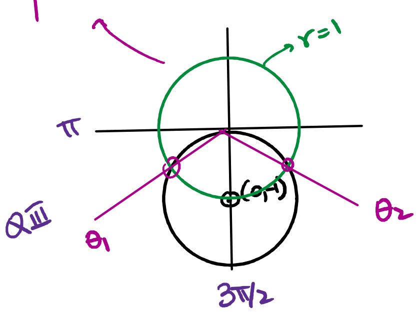


16. Sketch the polar curve $r = -2\sin\theta$.
b) State the bounds of integration to find the area inside the curve in the 3^{rd} quadrant?
c) At what angles will the above curve intersect with the polar curve $r = 1$?



$r = -2\sin\theta \rightsquigarrow a\sin\theta,\ a<0$

radius $= 1$, center : $(0,-1)$



b) bounds of integration

QIII: $\pi \leq \theta \leq 3\pi/2$

QIV: $-\pi/2 \leq \theta \leq 0$

c) Angles are θ_1 & θ_2

Equate r for both curves.

QI: $\sin(\pi/6) = 1/2$

$r = -2\sin\theta = 1$

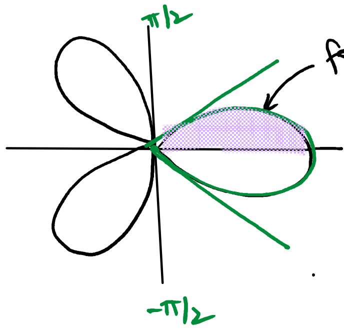
$\sin\theta = -1/2$

QIII: $\theta_1 = 7\pi/6$

QIV: $\theta_2 = \frac{11\pi}{6}$

$$A = \frac{1}{2}\int_{\theta=a}^{\theta=b} f(\theta)^2\, d\theta \qquad r = f(\theta)$$

17. a) Find the area of the region enclosed by one loop of the curve $r = 4\cos 3\theta$. → # petals = 3
b) Can you use symmetry in this case? c) What about for one loop of the curve $r = 4\sin 3\theta$?



Solve for a & b (bounds)

at (0,0), r=0, $\boxed{\text{solve for } \theta, \text{ when } r=0.}$

$4\cos(3\theta) = 0$

$\cos(3\theta) = 0$

$3\theta = -\pi/2,\ \pi/2$

$\therefore \theta = -\frac{\pi}{6}, \frac{\pi}{6}$

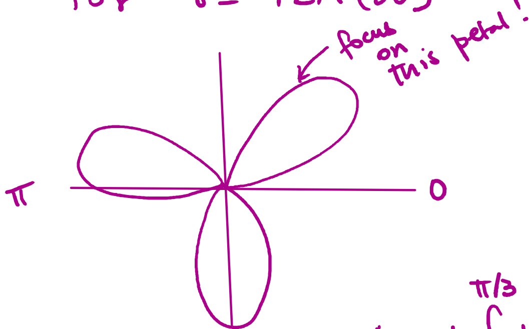
$$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} (4\cos 3\theta)^2\, d\theta$$

Using symmetry, $A = \cancel{2}\cdot\frac{1}{\cancel{2}}\int_0^{\pi/6} 16\cdot\cos^2(3\theta)\, d\theta = 16\int_0^{\pi/6} \frac{1+\cos(6\theta)}{2}\, d\theta$

$$= 8\left[\theta + \frac{\sin(6\theta)}{6}\Big|_0^{\pi/6}\right] = 8\left[\left(\frac{\pi}{6}-0\right) + \frac{1}{6}\left(\cancel{\sin\pi - \sin 0}^{\,0}\right)\right]$$

$$= \frac{8\pi}{6} = \frac{4\pi}{3} \text{ Ans.}$$

For $r = 4\sin(3\theta)$



$6\cdot\frac{\pi}{3} = 2\pi$

Solve for θ bounds when r=0.

$4\sin(3\theta) = 0$

$\sin(3\theta) = 0$ if $3\theta = 0, \pi$

$\theta = 0, \pi/3$

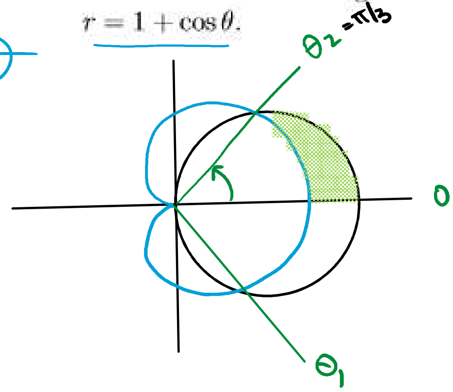
(Note: No symmetry used here)

$$A = \frac{1}{2}\int_0^{\pi/3} 16\sin^2(3\theta)\, d\theta = 8\int_0^{\pi/3} 1 - \frac{\cos(6\theta)}{2}\, d\theta$$

$$= 4\left[\theta - \frac{\sin(6\theta)}{6}\Big|_0^{\pi/3}\right] = 4\left[\left(\frac{\pi}{3}-0\right) - \frac{1}{6}\left(\cancel{\sin 2\pi - \sin 0}^{\,0}\right)\right]$$

$A = \frac{4\pi}{3}$ Ans.

18. Find the area of the region inside the curve $r = 3\cos\theta$ and outside the curve $r = 1 + \cos\theta$.



Use symmetry so that instead of

$\theta_1 \le \theta \le \theta_2$

we get $2\,[0 \le \theta \le \theta_2]$.

θ_2 in QI equate r

$3\cos\theta = 1 + \cos\theta$

$2\cos\theta = 1$

$\cos\theta = 1/2 \quad \therefore \theta = \theta_2 = \pi/3$

$$A = \cancel{2}\cdot\frac{1}{\cancel{2}}\int_0^{\pi/3} \Big[\underbrace{(3\cos\theta)^2 - (1+\cos\theta)^2}_{9\cos^2\theta - (1 + 2\cos\theta + \cos^2\theta)}\Big]\,d\theta$$

$$\underbrace{8\cos^2\theta}\ \underbrace{-1 - 2\cos\theta}$$

$$A = \overset{4}{\cancel{8}}\int_0^{\pi/3} \frac{1+\cos(2\theta)}{\cancel{2}}\,d\theta - \int_0^{\pi/3} 1\,d\theta - 2\int_0^{\pi/3}\cos\theta\,d\theta$$

$$= 4\left[\theta + \frac{\sin(2\theta)}{2}\Big|_0^{\pi/3}\right] - \theta\Big|_0^{\pi/3} - 2\sin\theta\Big|_0^{\pi/3}$$

$$= 4\left(\frac{\pi}{3}\right) + 2\left(\sin\left(\frac{2\pi}{3}\right) - \cancel{\sin(0)}^{\,0}\right) - \left(\frac{\pi}{3} - 0\right) - 2\left(\sin\frac{\pi}{3} - \cancel{\sin 0}^{\,0}\right)$$

$$= \frac{4\pi}{3} + \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}} - \frac{\pi}{3} - \cancel{2}\frac{\sqrt{3}}{\cancel{2}}$$

$$= \frac{4\pi}{3} - \frac{\pi}{3} + \cancel{\sqrt{3}} - \cancel{\sqrt{3}}$$

$$= \frac{3\pi}{3} = \pi$$

Ans.