WIR#9

MATHEMATICS
TEXAS A&M UNIVERSITY

Week-in-Review#9 - Sections 4.1 and 4.2
Math 142 - Fall 2024, by K. Kilmer

Section 4.1

- The **General Antiderivative** of $f(x)$ on an interval is $F(x)+C$, where C is any real number constant, if $\frac{d}{dx}(F(x)+C)=f(x)$.
- The collection of all antiderivatives of a function, $f(x)$, is called the **indefinite integral**, and is denoted by $\int f(x)\,dx$ (the indefinite integral of $f(x)$ with respect to x). If we know one function $F(x)$ for which $F'(x)=f(x)$, then $\int f(x)\,dx = F(x)+C$.
- **Rules of Integration:**
  - $\int k\,dx = kx + C$, where k is any real number
  - $\int x^n\,dx = \frac{1}{n+1}x^{n+1}+C$, where n is any real number with $\underline{n \neq -1}$
    - ~~$\int x^{-1}dx = \frac{1}{-1+1}x^{-1+1}$~~ we can't divide by zero
  - $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x|+C$
  - → $\int b^x\,dx = \frac{1}{\ln b}\cdot b^x + C$ where b is any positive real number
  - $\int e^x\,dx = e^x + C$
  - → $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
  - → $\int k\cdot f(x)\,dx = k\int f(x)\,dx$

1. Evaluate the following:

(a) $\int x^4\,dx = \frac{1}{4+1}x^{4+1}+C = \boxed{\frac{1}{5}x^5+C}$ ✓

Check:

$$\frac{d}{dx}\left(\frac{1}{5}x^5+C\right)$$

$$= \frac{1}{\cancel{5}}\cdot 5x^4 + 0 = x^4 ✓$$

$(x^3)^{1/2} = x^{3/2}$

(b) $\int \left(\frac{1}{5}e^x + 6\sqrt[2]{x^3}\right)\,dx$

$$= \int\left(\underbrace{\frac{1}{5}e^x}+\underbrace{6x^{3/2}}\right)dx ✓ = \int \frac{1}{5}e^x dx + \int 6x^{3/2}dx$$

$$= \frac{1}{5}\int e^x dx + 6\int x^{3/2}dx$$

$$= \frac{1}{5}e^x + 6\cdot\frac{1}{\frac{3}{2}+1}x^{\frac{3}{2}+1} + C = \frac{1}{5}e^x + 6\cdot\frac{2}{5}x^{5/2} + C = \boxed{\frac{1}{5}e^x + \frac{12}{5}x^{5/2} + C}$$

Check:

$$\frac{d}{dx}\left(\underbrace{\frac{1}{5}e^x}+\underbrace{\frac{12}{5}x^{5/2}}+\underbrace{C}\right)$$

$$= \frac{1}{5}e^x + \frac{12}{\cancel{5}}\cdot\frac{\cancel{5}}{2}x^{3/2} + 0$$

$$= \frac{1}{5}e^x + 6x^{3/2} ✓$$

$$\frac{1}{\sqrt[4]{x^3}} = \frac{1}{x^{3/4}} = x^{-3/4}$$

(c) $\int \left(\frac{4}{x^7} - 5x^{-1} - \frac{1}{\sqrt[4]{x^3}} + 3x^{-3} + 7^x - 8 \right) dx$

$$= \int \left(\underbrace{4x^{-7}}_{} \underbrace{-5 \cdot \frac{1}{x}}_{} - \underbrace{x^{-3/4}}_{} + \underbrace{3x^{-3}}_{} + \underbrace{7^x}_{} - \underbrace{8}_{} \right) dx$$

$$= 4 \cdot \frac{1}{-7+1} x^{-7+1} - 5 \cdot \ln|x| - \frac{1}{-\frac{3}{4}+1} x^{-\frac{3}{4}+1} + 3 \cdot \frac{1}{-3+1} x^{-3+1} + \frac{1}{\ln 7} \cdot 7^x - 8x + C$$

$$= 4 \cdot \frac{1}{-6} x^{-6} - 5\ln|x| - \frac{1}{\frac{1}{4}} x^{1/4} + 3 \cdot \frac{1}{-2} x^{-2} + \frac{1}{\ln 7} \cdot 7^x - 8x + C$$

$$= \boxed{-\frac{2}{3} x^{-6} - 5\ln|x| - 4x^{1/4} - \frac{3}{2} x^{-2} + \frac{1}{\ln 7} \cdot 7^x - 8x + C}$$

(d) $\int \frac{4\sqrt{x} + 3x^7 - 4}{2\sqrt[5]{x^2}} \, dx$ $= \int \left(\frac{4\sqrt{x}}{2\sqrt[5]{x^2}} + \frac{3x^7}{2\sqrt[5]{x^2}} - \frac{4}{2\sqrt[5]{x^2}} \right) dx$

$$= \int \left(2 \cdot \frac{x^{1/2}}{x^{2/5}} + \frac{3}{2} \cdot \frac{x^7}{x^{2/5}} - 2 \cdot \frac{1}{x^{2/5}} \right) dx$$

$$= \int \left(2x^{\frac{1}{2}-\frac{2}{5}} + \frac{3}{2} x^{7-\frac{2}{5}} - 2 \cdot x^{-2/5} \right) dx$$

$$\frac{3}{2} \cdot \frac{5}{38} = \frac{15}{76}$$

$$= \int \left(\underbrace{2x^{\frac{1}{10}}}_{} + \underbrace{\frac{3}{2} x^{\frac{33}{5}}}_{} - \underbrace{2x^{-2/5}}_{} \right) dx$$

$$= 2 \cdot \frac{1}{\frac{1}{10}+1} x^{\frac{1}{10}+1} + \frac{3}{2} \cdot \frac{1}{\frac{33}{5}+1} x^{\frac{33}{5}+1} - 2 \cdot \frac{1}{-\frac{2}{5}+1} x^{-\frac{2}{5}+1} + C$$

$$= 2 \cdot \frac{1}{\frac{11}{10}} x^{11/10} + \frac{3}{2} \cdot \frac{1}{\frac{38}{5}} x^{38/5} - 2 \cdot \frac{1}{3/5} x^{3/5} + C$$

$$= \boxed{\frac{20}{11} x^{11/10} + \frac{15}{76} x^{38/5} - \frac{10}{3} x^{3/5} + C}$$

(e) $\int (4x^2 + 7)(7x - 9x^4) \, dx$

$$= \int \left(\underbrace{28x^3}_{} \underbrace{-36x^6}_{} + \underbrace{49x^1}_{} - \underbrace{63x^4}_{} \right) dx$$

$$= 28 \cdot \frac{1}{3+1} x^{3+1} - 36 \cdot \frac{1}{6+1} x^{6+1} + 49 \cdot \frac{1}{1+1} x^{1+1} - 63 \cdot \frac{1}{4+1} x^{4+1} + C$$

$$= \boxed{7x^4 - \frac{36}{7} x^7 + \frac{49}{2} x^2 - \frac{63}{5} x^5 + C}$$

2. Find $f(x)$ if $f'(x) = \dfrac{3e^{-2x}+4e^{-x}}{2e^{-2x}}$ and $f(0) = 5$.

we are being asked to find a specific antiderivative

① Find the general antiderivative

$$f(x) = \int f'(x)\,dx = \int \frac{3e^{-2x}+4e^{-x}}{2e^{-2x}}\,dx = \int\left(\frac{3\cancel{e^{-2x}}}{2\cancel{e^{-2x}}} + \frac{4e^{-x}}{2e^{-2x}}\right)dx$$

$$= \int\left(\frac{3}{2}+2e^{-x-(-2x)}\right)dx = \int\left(\underline{\frac{3}{2}}+\underline{2e^{x}}\right)dx = \frac{3}{2}x+2e^{x}+C$$

② use $f(0)=5$ to solve for C:

$f(x) = \frac{3}{2}x+2e^{x}+C$

$5 = \frac{3}{2}(0)+2e^{0}+C$

$5 = 2+C$

$3 = C$

③ Plug in C to $f(x)$

$$\boxed{f(x) = \frac{3}{2}x+2e^{x}+3}$$

3. Find the cost of producing 10 items if the marginal cost, in dollars per item, is given by $f(x) = 150 - 0.01e^{x}$, where x is the number of items produced. Assume the fixed costs are \$100.

$C(0) = 100$

We need $C(10)$.

① Find the general antiderivative

$C(x) = \int(150-0.01e^{x})\,dx = 150x-0.01e^{x}+K$

② Solve for K using $C(0)=100$

$C(x) = 150x-0.01e^{x}+K$

$100 = 150(0)-0.01e^{0}+K$

$100 = -0.01+K$

$100.01 = K$

③ Plug K into $C(x)$

$C(x) = 150x-0.01e^{x}+100.01$

④ Find $C(10)$

$C(10) = 150(10)-0.01e^{10}+100.01$

$\approx \boxed{\$1379.75}$

Section 4.2

- We use u-substitution when our integrand is the result (or nearly the result) of the Chain Rule. We follow the process outlined below:
  - Select u (look for function of x where you normally have just x)

    $\int u^{n}\cdot$ (other stuff) dx OR $\int b^{u}\cdot$ (other stuff) dx OR $\int \frac{1}{u}\cdot$ (other stuff) dx
  - Take the derivative of u using $\frac{du}{dx}$ notation.
  - Bring dx to the right hand side.
  - Bring any constant multiples to the left-hand side.
  - Substitute to replace all terms with x's.
  - Integrate with u's.
  - Return x's into the problem.

① $\int (u)^n \cdot \underline{\quad}$ ② $\int b^{(u)} \cdot \underline{\quad}$ ③ $\int \frac{1}{(u)} \cdot \underline{\quad}$

4. Evaluate the following integrals:

(a) $\int \frac{8x+21x^2}{4x^2+7x^3}\,dx$

$$= \int \underbrace{\frac{1}{4x^2+7x^3}}_{u} \cdot \underbrace{(8x+21x^2)\,dx}_{du}$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \boxed{\ln|4x^2+7x^3| + C}$$

$\rightarrow u = 4x^2+7x^3$

$\frac{du}{dx} = 8x+21x^2$

$\rightarrow du = \underbrace{(8x+21x^2)\,dx}$

Check:

$$\frac{d}{dx}\left(\ln|\underbrace{4x^2+7x^3}|+C\right)$$

$$= \frac{1}{4x^2+7x^3} \cdot (8x+21x^2) + 0$$

$$= \frac{8x+21x^2}{4x^2+7x^3} ✓$$

(b) $\int \frac{20x^7-15x}{(3x^8-9x^2)^{13}}\,dx$

$$= \int (\underbrace{3x^8-9x^2}_{u})^{-13}(20x^7-15x)\,dx$$

$$= \int (\underbrace{3x^8-9x^2}_{u})^{-13} \cdot 5\underbrace{(4x^7-3x)\,dx}_{\frac{1}{6}du}$$

$$= \int u^{-13} \cdot 5 \cdot \frac{1}{6}\,du$$

$$= \frac{5}{6}\int u^{-13}\,du$$

$$= \frac{5}{6} \cdot \frac{1}{-13+1} u^{-13+1} + C = \frac{5}{-72} u^{-12} + C$$

$$= \boxed{\frac{-5}{72}(3x^8-9x^2)^{-12} + C}$$

$u = \underbrace{3x^8-9x^2}$

$du = (24x^7-18x)\,dx$

$\frac{du}{6} = \frac{\cancel{6}(4x^7-3x)\,dx}{\cancel{6}}$

$\frac{1}{6}du = \underbrace{(4x^7-3x)\,dx}$

(c) $\int \frac{64x^3-32x^7+2e^x}{\sqrt[7]{(8x^4-2x^8+e^x)^5}}\,dx$

$$= \int \frac{64x^3-32x^7+2e^x}{(8x^4-2x^8+e^x)^{5/7}}\,dx = \int (\underbrace{8x^4-2x^8+e^x}_{u})^{-5/7}(64x^3-32x^7+2e^x)\,dx$$

$$= \int (\underbrace{8x^4-2x^8+e^x}_{u})^{-5/7} \cdot 2\underbrace{(32x^3-16x^7+e^x)\,dx}_{du}$$

$$= \int u^{-5/7} \cdot 2\,du$$

$$= 2\int u^{-5/7}\,du$$

$$= 2 \cdot \frac{1}{2/7} u^{2/7} + C = 7u^{2/7} + C = \boxed{7(8x^4-2x^8+e^x)^{2/7} + C}$$

$u = \underbrace{8x^4-2x^8+e^x}$

$du = \underbrace{(32x^3-16x^7+e^x)\,dx}$

$2 \cdot \frac{1}{2/7} = \cancel{2} \cdot \frac{7}{\cancel{2}} = 7$

$-\frac{6}{x^2} = -6x^{-2}$

① $\int (u)^n \cdot \underline{\qquad}$ ② $\int b^{(u)} \cdot \underline{\qquad}$ ③ $\int \frac{1}{(u)} \cdot \underline{\qquad}$

(d) $\int \frac{4 \cdot 2^{-6/x^2}}{x^3}\,dx$ $= \int 2^{\underbrace{-6x^{-2}}_{u}} \cdot 4\underbrace{x^{-3}\,dx}_{\frac{1}{12}du}$

$u = -6x^{-2}$

$\frac{du}{12} = \frac{\cancel{12}x^{-3}\,dx}{\cancel{12}}$

$\frac{1}{12}du = \underbrace{x^{-3}\,dx}$

$= \int 2^u \cdot 4 \cdot \frac{1}{12}\,du$

$= \frac{1}{3}\int 2^u\,du = \frac{1}{3} \cdot \frac{1}{\ln 2} \cdot 2^u + C$

$= \boxed{\frac{1}{3\ln 2} \cdot 2^{-6x^{-2}} + C}$

(e) $\int \frac{5\sqrt{\ln x}}{3x}\,dx$ $= \int \frac{5(\ln x)^{1/2}}{3x}\,dx = \int (\underbrace{\ln x}_{u})^{1/2} \cdot \frac{5}{3} \cdot \underbrace{\frac{1}{x}\,dx}_{du}$

$u = \ln x$

$du = \underbrace{\frac{1}{x}\,dx}$

$= \int u^{1/2} \cdot \frac{5}{3}\,du$

$= \frac{5}{3}\int u^{1/2}\,du$

$= \frac{5}{3} \cdot \frac{2}{3}u^{3/2} + C$

$= \frac{10}{9}u^{3/2} + C$

$= \boxed{\frac{10}{9}(\ln x)^{3/2} + C}$

5. Find $f(x)$ if $f'(x) = (10x + 45)\sqrt[3]{x^2 + 9x + 27}$ and $f(0) = 310$.

$u = \underbrace{x^2+9x+27}$

$du = \underbrace{(2x+9)\,dx}$

①Find the general antiderivative:

$f(x) = \int (10x+45)\sqrt[3]{x^2+9x+27}\,dx = \int (\underbrace{x^2+9x+27}_{u})^{1/3}(10x+45)\,dx$

$= \int (\underbrace{x^2+9x+27}_{u})^{1/3} \cdot 5\underbrace{(2x+9)\,dx}_{du}$

$= \int u^{1/3} \cdot 5\,du$

$= 5\int u^{1/3}\,du = 5 \cdot \frac{3}{4}u^{4/3} + C = \frac{15}{4}(x^2+9x+27)^{4/3} + C$

② Use $f(0) = 310$ to solve for C:

$\underset{\substack{\uparrow\\310}}{f(x)} = \frac{15}{4}(\underset{\substack{\uparrow\\0}}{x}^2 + 9\underset{\substack{\uparrow\\0}}{x} + 27)^{4/3} + C$

$310 = \frac{15}{4}(0^2 + 9(0) + 27)^{4/3} + C$

$310 = \frac{15}{4}(27)^{4/3} + C$

$310 = 303.75 + C$

$6.25 = C$

③Plug in C

$\boxed{f(x) = \frac{15}{4}(x^2+9x+27)^{4/3} + 6.25}$