# Math 140: Week-in-Review 5 (3.4 & 4.1)

1. objective: max
2. $x \geq 0, y \geq 0$, non-negative
3. constraints: $ax+by+\cdots \leq c$, $c \geq 0$

1. Determine if the following linear programming problems are standard maximization problems. If they are, transform the constraints of the linear programming problems to linear equations with slack variables and write down the corresponding simplex tableau.

(a)

Objective: Maximize $P = 2x + 5y$
Subject to: $5y \leq -2x + 20$
$3 - 4y \leq x$
$x \geq 0, \ y \geq 0.$

1. max ✓
2. $x \geq 0, y \geq 0$ ✓
3. (rewrite the constraints)

$2x + 5y \leq 20$ ✓

$-x - 4y \leq -3$ ✗

SOLN: NOT A STANDARD MAXIMIZATION PROBLEM

(b)

Objective: Maximize $P = 7x + 3y + 5z$
Subject to: $x + y + z \leq 20$
$-3x + 4y \geq -18 + 5z$
$2x \leq -3z + 30$
$x \geq 0, \ y \geq 0, \ z \geq 0.$

1. max ✓
2. $x \geq 0, y \geq 0$ ✓
3. $-3x + 4y - 5z \geq -18$

$3x - 4y - 5z \leq 18$ ✓

$2x + 3z \leq 30$ ✓

SOLN: STANDARD MAXIMIZATION PROBLEM

2. For the following simplex tableau, identify the pivot column, pivot row, and the pivot element to be used in the next iteration of the Simplex Method. Show supporting work.

(a)

| $x$ | $y$ | $s_1$ | $s_2$ | $s_3$ | $P$ | constant | |
|---|---|---|---|---|---|---|---|
| 3 | 1 | 1 | 0 | 0 | 0 | 24 | $\frac{24}{1}$ |
| 2 | 0 | 0 | 1 | 0 | 0 | 18 | ✗ (undefined ratio) |
| 1 | 3 | 0 | 0 | 1 | 0 | 24 | $\frac{24}{3}$ → smaller ⇒ pivot row |
| −4 | −6 | 0 | 0 | 0 | 1 | 24 | |

pivot row

↑ pivot column (most negative entry in the bottom row)

The pivot element is (3) in Row 3 Column 2

(b)

| $x$ | $y$ | $z$ | $s_1$ | $s_2$ | $s_3$ | $P$ | constant | |
|---|---|---|---|---|---|---|---|---|
| −1/2 | 0 | 1/4 | 1 | −1/4 | 0 | 0 | 17/2 | ✗ (negative ratio) |
| 1/2 | 1 | 3/4 | 0 | 1/4 | 0 | 0 | 19/2 | $(\frac{19}{2})/(\frac{1}{2}) = 19$ |
| 2 | 0 | 3 | 0 | 0 | 1 | 0 | 30 | $\frac{30}{2} = 15$ (smaller) ⇒ pivot row |
| −1 | 0 | −1/2 | 6 | 3/2 | 0 | 1 | 63 | |

pivot row

↑ pivot column

The pivot element is (2) in Row 3 Column 1

3. Read the values of each variable from the following tableau below and classify each variable as basic or non-basic. What corner point does each tableau correspond to, if the problem had been solved using the Method of Corners? Determine if the solution is optimal and explain why.

(a)

| $x$ | $y$ | $s_1$ | $s_2$ | $s_3$ | $P$ | constant |
|---|---|---|---|---|---|---|
| 3 | 0 | 1 | 0 | −1 | 0 | 3 |
| −7 | 0 | 0 | 1 | 3 | 0 | 5 |
| −2 | 1 | 0 | 0 | 1 | 0 | 2 |
| −4 | 0 | 0 | 0 | 3 | 1 | 6 |

Basic variables: the only non-zero entry in their column is a single 1

Non-basic variables: not basic. Set all non-basic variables to zero

Basic
$y = 2$
$s_1 = 3$
$s_2 = 5$
$P = 6$

Non-basic
$x = 0$
$s_3 = 0$

Corner pnt
$(x, y) = (0, 2)$

SOLN is not optimal because the last row has a negative entry −4

(b)

| $x$ | $y$ | $s_1$ | $s_2$ | $s_3$ | $P$ | constant |
|---|---|---|---|---|---|---|
| 0 | 0 | 7/3 | 1 | 2/3 | 0 | 4 |
| 1 | 0 | 0 | −1/7 | −3/7 | 0 | 3/7 |
| 0 | 1 | 4/7 | −2/7 | 1/7 | 0 | 20/7 |
| 0 | 0 | 0 | −4/7 | 9/7 | 1 | 54/7 |

Basic
$x = 3/7$
$y = 20/7$
$P = 54/7$

Non-basic
$s_1 = 0$
$s_2 = 0$
$s_3 = 0$

Corner pnt
$(x, y) = (3/7, 20/7)$

SOLN is NOT optimal because the last row has a negative entry $-4/7$

(c)

| $x$ | $y$ | $s_1$ | $s_2$ | $s_3$ | $P$ | constant |
|---|---|---|---|---|---|---|
| 0 | 0 | 7/3 | 1 | 2/3 | 0 | 4 |
| 1 | 0 | 1/3 | 0 | −1/3 | 0 | 1 |
| 0 | 1 | 2/3 | 0 | 1/3 | 0 | 4 |
| 0 | 0 | 4/3 | 0 | 5/3 | 1 | 10 |

Basic
$x = 1$
$y = 4$
$s_2 = 4$

Non-basic
$s_1 = 0$
$s_3 = 0$

Corner pnt
$(x, y) = (1, 4)$

SOLN is OPTIMAL because the last row contains only non-negative entries.

4. Solve the following linear programming problem using the Simplex Method.

Objective: Maximize $P = 2x + 3y$
Subject to: $5x + 4y \leq 32 \Rightarrow 5x + 4y + s_1 = 32$
$x + 2y \leq 10 \Rightarrow x + s_2 = 8$
$x \geq 0,\ y \geq 0.$
$-2x - 3y + P = 0$

| $x$ | $y$ | $s_1$ | $s_2$ | $P$ | const. |
|---|---|---|---|---|---|
| 5 | 4 | 1 | 0 | 0 | 32 |
| 1 | 2 | 0 | 1 | 0 | 10 |
| −2 | −3 | 0 | 0 | 1 | 0 |

pivot row; pivot column

$\frac{32}{4} = 8$

$\frac{10}{2} = 5$ (smaller)

pivot on 2 in Row 2 Col 2

| $x$ | $y$ | $s_1$ | $s_2$ | $P$ | const. |
|---|---|---|---|---|---|
| 3 | 0 | 1 | −2 | 0 | 12 |
| $\frac{1}{2}$ | 1 | 0 | $\frac{1}{2}$ | 0 | 5 |
| $-\frac{1}{2}$ | 0 | 0 | $\frac{3}{2}$ | 1 | 15 |

pivot row; pivot column

$\frac{12}{3} = 4$ (smaller)

$\frac{5}{(\frac{1}{2})} = 10$

pivot on 3 in Row 1 Column 1

| $x$ | $y$ | $s_1$ | $s_2$ | $P$ | const. |
|---|---|---|---|---|---|
| 1 | 0 | $\frac{1}{3}$ | $-\frac{2}{3}$ | 0 | 4 |
| 0 | 1 | $-\frac{1}{6}$ | $\frac{5}{6}$ | 0 | 3 |
| 0 | 0 | $\frac{1}{6}$ | $\frac{7}{6}$ | 1 | 17 |

→ all non-negative entries
⇒ final tableau

| Basic | Non-basic |
|---|---|
| $x = 4$ | $s_1 = 0$ |
| $y = 3$ | $s_2 = 0$ |
| $P = 17$ | |

SOLN: max $P = 17$ at the point $(x, y) = (4, 3)$

5. **Solve the following linear programming problem using the Simplex Method.**

A farmer owns a 12 acre piece of land and cultivates greens, carrots, and cucumbers. Each acre of crop that is planted has certain requirements for labor and capital given in the table below.

| | Capital ($) | Labor (hrs) | Profit ($) |
|---|---|---|---|
| Greens (per acre) | 36 | 6 | 40 |
| Carrots (per acre) | 24 | 6 | 30 |
| Cucumbers (per acre) | 18 | 2 | 20 |

If the farmer has $360 available for capital and 48 hours of labor for cultivating these crops, how many acres of each crop should be planted to maximize profit? What is the maximum profit? Is there any leftover land, capital, or labor-hours?

Let $x$ = number of acres of greens cultivated
$y$ = number of acres of carrots cultivated
$z$ = number of acres of cucumbers cultivated
$P$ = profit made from sale of these vegetables

Objective: MAX $P = 40x + 30y + 20z$

Subject to: (land) $x + y + z \leq 12$
(capital) $36x + 24y + 18z \leq 360$
(labor-hrs) $6x + 6y + 2z \leq 48$
(non-negative) $x \geq 0, y \geq 0, z \geq 0$

STANDARD MAX. PROBLEM ⇓ can use SIMPLEX METHOD

Introduce slack variables ($s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$) for each constraint (except for non-neg.)

$$\begin{cases} x + y + z + s_1 = 12 \\ 36x + 24y + 18z + s_2 = 360 \\ 6x + 6y + 2z + s_3 = 48 \\ -40x - 30y - 20z + P = 0 \end{cases}$$

constraints; objective

| x | y | z | $s_1$ | $s_2$ | $s_3$ | P | const |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 12 |
| 36 | 24 | 18 | 0 | 1 | 0 | 0 | 360 |
| 6 | 6 | 2 | 0 | 0 | 1 | 0 | 48 |
| -40 | -30 | -20 | 0 | 0 | 0 | 1 | 0 |

$\frac{12}{1} = 12$

$\frac{360}{36} = 10$

$\frac{48}{6} = 8$ (smallest) pivot row

pivot on 6 in Row 3 Column 1

pivot column

| x | y | z | $s_1$ | $s_2$ | $s_3$ | P | const |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 2/3 | 1 | 0 | -1/6 | 0 | 4 |
| 0 | -12 | 6 | 0 | 1 | -6 | 0 | 72 |
| 1 | 1 | 1/3 | 0 | 0 | 1/6 | 0 | 8 |
| 0 | 10 | -20/3 | 0 | 0 | 20/3 | 1 | 320 |

$4/(2/3) = 6$ (smallest)

$\frac{72}{6} = 12$

$8/(\frac{1}{3}) = 24$

pivot on 2/3 in Row 1 Column 3

pivot column

| x | y | z | $s_1$ | $s_2$ | $s_3$ | P | const |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 3/2 | 0 | -1/4 | 0 | 6 |
| 0 | -12 | 0 | -9 | 1 | -9/2 | 0 | 36 |
| 1 | 1 | 0 | -1/2 | 0 | 1/4 | 0 | 6 |
| 0 | 10 | 0 | 10 | 0 | 5 | 1 | 360 |

| Basic | Non-basic |
|---|---|
| $x = 6$ | $y = 0$ |
| $z = 6$ | $s_1 = 0$ |
| $s_2 = 36$ | $s_3 = 0$ |
| $P = 360$ | |

* no negative entries in the final row ⇒ final tableau

SOLN: The farmer must cultivate 6 acres of greens, 0 acres of carrots, and 6 acres of cucumbers to generate a maximum profit of $360. There will be no leftover land or labor-hours, but $36 of unused capital remains.

6. Consider an experiment where you select a letter at random from the word "chance".

(a) State the sample space for this experiment.

$S = \{c, h, a, n, e\}$

* the two c's are indistinguishable *

(b) State all simple events for the experiment.

↓ only one outcome

$\{c\}, \{h\}, \{a\}, \{n\}, \{e\}$

(c) Give an example of an impossible event for the experiment.

↳ an event that can't happen

E = "select the letter k from the word chance"

$E = \{\} = \phi$ (empty)

(d) State the total number of possible events.

$n = 5$ (number of outcomes)

* 32 subsets of S *

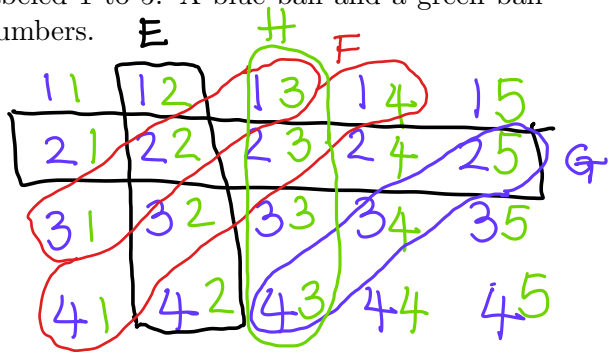
Total number of events $= 2^5 = 32$

letters a, e, i, o, u

(e) Write the event, $V$, "a vowel is selected".

$V = \{a, e\}$

7. You have two bowls containing pingpong balls. One bowl contains 4 identical blue balls labeled 1 to 4. The other bowl contains 5 identical green balls labeled 1 to 5. A blue ball and a green ball are drawn at random, one from each bowl, noting the numbers.




(a) What is the sample space for this experiment?

$$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),$$
$$(2,1),(2,2),(2,3),(2,4),(2,5),$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),$$
$$(4,1),(4,2),(4,3),(4,4),(4,5)\}$$

number of outcomes $= 4 \times 5 = 20$

(b) Write the event, E, that a ball labeled 2 is drawn.

$$E = \{(1,2),(2,1),(2,2),(2,3),(2,4),(2,5),(3,2),(4,2)\}$$

(c) Write the event, F, that a sum of 4 or a sum of 5 is drawn.

combine

$$F = \{(1,3),(2,2),(3,1),(4,1),(3,2),(2,3),(1,4)\}$$

sum of 4 | sum of 5

(d) Write the event, G, that a sum of 7 is drawn.

$$G = \{(2,5),(3,4),(4,3)\}$$

(e) Write the event, H, that the number on the green ball is a 3.

$H = \{(1,3),(2,3),(3,3),(4,3)\}$

(f) Verbally describe the event $H^C$

$H^C$ = the event "a number other than 3 is drawn on the green ball"

(g) Are the events $G$ and $H$ mutually exclusive? Explain why or why not.

$G \cap H = \{(4,3)\}$ is not empty, so

G and H are not mutually exclusive.

8. A fair coin is tossed three times, noting the side landing up on each toss.

(a) State the sample space for this experiment.

$S = \{HHH, THH, HTH, HHT, TTH, THT, HTT, TTT\}$

(b) State the event, E, that the second toss is "tails".

$E = \{(HTH), (TTH), (HTT), (TTT)\}$

(c) State the event, F, that all tosses land on "heads".

$F = \{(HHH)\}$

(d) State the event $E \cup F$. → combine outcomes

$E \cup F = \{(HTH), (TTH), (HTT), (TTT), (HHH)\}$

(e) State the event $E \cap F$. → common outcomes

$E \cap F = \{\} = \phi$ * no outcomes in common

(f) Are events E and F mutually exclusive? Explain why or wy not.

since $E \cap F = \{\} = \phi$, E and F are mutually exclusive events

9. A card is drawn at random from a standard 52-card deck, noting the suit drawn, and a ball is drawn at random from a bowl containing three balls labeled 1 to 3.

Hearts, Diamonds, Spades, Clubs

(a) State the number of outcomes in the sample space, $S$, of this experiment.

# of outcomes = $4 \times 3 = 12$

$$S = \{(H,1),(H,2),(H,3),(D,1),(D,2),(D,3),(D,4),(S,1),(S,2),(S,3),(C,1),(C,2),(C,3),(C,4)\}$$

(b) State the number of simple events of the experiment. Write one.

* simple events are subsets of S with only one outcome

S has 12 outcomes $\Rightarrow$ the number of simple events is 12

(c) State the event, $E$, that a ball labeled 2 is drawn from the bowl.

$$E = \{(H,2),(D,2),(S,2),(C,2)\}$$

(d) State the event, $F$, that a red card is drawn and a number less than 3 is drawn.

H, D — both — 1, 2

$$F = \{(H,1),(H,2),(D,1),(D,2)\}$$

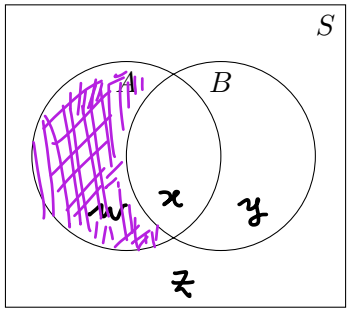
(e) State the event, $G$, that a Spade is drawn or a number greater than 1 is drawn from the bowl.

one or the other or both

$$G = \{(S,1),(S,2),(S,3),(H,2),(H,3),(D,2),(D,3),(C,2),(C,3)\}$$

10. Shade the region on the Venn diagram representing the events given below.

**a.** $B^C \cap A$



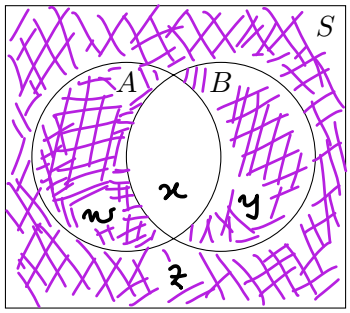

$S = \{x, y, z, w\}$ $A = \{w, x\}$ $B = \{x, y\}$

$B^C = \{w, z\}$

$B^C \cap A = \{w\}$

* S contains everything, not just z! *

**b.** $(A \cap B)^C$




$S = \{w, x, y, z\}$

$A \cap B = \{x\}$

$(A \cap B)^C = \{w, y, z\}$

**c.** $(A^C \cup B)^C$

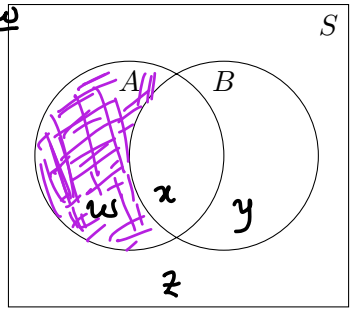
DeMorgan's Law

$(A^C \cup B)^C$

$= (A^C)^C \cap B^C$

$= A \cap B^C$

↗ same as (a)




$S = \{w, x, y, z\}$

$A^C = \{y, z\}$, $B = \{x, y\}$

$A^C \cup B = \{y, z, x\}$

$(A^C \cup B)^C = \{w\}$

11. A letter is selected at random from the word "chance," noting the letter, and then a spinner divided into three equal regions (maroon, white, and gray) is spun, noting the color. If the spinner lands on a dividing line, it is re-spun. Consider the following events.

$W :=$ the event "a consonant is drawn". $= \{c, h, n\}$
$X :=$ the event, "an h is drawn". $= \{h\}$
$Y :=$ the event "the color spun is maroon". $\{$ maroon $\}$

(a) Verbally describe the outcomes in each of the folowing events.

(i) $W^C$

$W^C$ = the event "a vowel is drawn"

(ii) $W \cap Y^C$ (AND)

$W \cap Y^C$ = the event "a consonant is drawn AND a color other than maroon is spun"

(iii) $Y^C \cup (W \cap X)$ (OR, AND) $\quad W \cap X = \{h\}$

$Y^C \cup (W \cap X)$ = the event "a color other than maroon is spun OR the letter "h" is drawn"

(b) Write each of the following events using symbolic notation.

(i) The event "a vowel is drawn and the color spun is maroon".

(and → intersection; vowel = not a consonant = $W^C$; $Y$)

$W^C \cap Y$

(ii) The event "a consonant other than "h" is drawn, and a color other than maroon is spun".

(and → intersection; $X^C$; $Y^C$)

$X^C \cap Y^C$