Math 151 - Week-In-Review 5

Topics for the week:

3.1 Derivatives of Polynomials and Exponential Functions
3.2 The Product and Quotient Rules

3.1 Derivatives of Polynomials and Exponential Functions

1. Compute the derivative of $f(x) = 3x^5 - 4x^2 + 6$.

$f(x) = 3x^5 - 4x^2 + 6$

$f'(x) = 15x^4 - 8x + 0$

$f'(x) = 15x^4 - 8x$

2. Compute $\dfrac{dg(t)}{dt}$ for $g(t) = \dfrac{1}{3}t^8 - \dfrac{2}{7t^4} + 6t^{-3}$.

$g(t) = \frac{1}{3}t^8 - \frac{2}{7}t^{-4} + 6t^{-3}$

$\frac{dg(t)}{dt} = \frac{8}{3}t^7 + \frac{8}{7}t^{-5} - 18t^{-4}$

3. For $y = \dfrac{1 + x - 4\sqrt{x}}{x}$, find $\dfrac{dy}{dx}$.

$y = \frac{1}{x} + \frac{x}{x} - \frac{4\sqrt{x}}{x} = 1x^{-1} + 1 - 4x^{-1/2}$

$\frac{dy}{dx} = -1x^{-2} + 0 + 2x^{-3/2}$

$\frac{dy}{dx} = \frac{-1}{x^2} + \frac{2}{\sqrt{x^3}}$

Note:

$\frac{4\sqrt{x}}{x} = \frac{4x^{1/2}}{x^1} = 4x^{-1/2}$

and

$x^{-3/2} = \frac{1}{x^{3/2}} = \frac{1}{\sqrt{x^3}}$

4. Compute the derivative of f with respect to t, $f(t) = \dfrac{7}{3x^2} - \dfrac{5}{2x} - \dfrac{3}{e^{-x}}$.

$f(t) = \frac{7}{3}x^{-2} - \frac{5}{2}x - 3e^x$

$f'(t) = \frac{-14}{3}x^{-3} - \frac{5}{2} - 3e^x$

5. Compute $f'(x)$ for $f(x) = 11e^x + e^{11}x$.

Note: e^{11} is a constant

$f(x) = 11e^x + e^{11}x$

$f'(x) = 11e^x + e^{11}$

6. Compute $g'(z)$ for $g(z) = \sqrt[5]{z} + 10\sqrt[6]{z^5}$.

$g(z) = z^{1/5} + 10z^{5/6}$

$g'(z) = \frac{1}{5}z^{-4/5} + \frac{50}{6}z^{-1/6}$

Remember when you subtract fractions to get a common denominator.

$g'(z) = \dfrac{1}{5z^{4/5}} + \dfrac{50}{6z^{1/6}}$

$g'(z) = \dfrac{1}{5\sqrt[5]{z^4}} + \dfrac{50}{6\sqrt[6]{z}}$

7. Given a position function of $s(t) = \dfrac{(t^3+1)(t^2-t+1)}{t^4}$, determine the corresponding velocity and acceleration functions.

$s(t) = \dfrac{t^5}{t^4} - \dfrac{t^4}{t^4} + \dfrac{t^3}{t^4} + \dfrac{t^2}{t^4} - \dfrac{t}{t^4} + \dfrac{1}{t^4} = t - 1 + t^{-1} + t^{-2} - t^{-3} + t^{-4}$

$s'(t) = v(t) = 1 - t^{-2} - 2t^{-3} + 3t^{-4} - 4t^{-5}$

$v'(t) = s''(t) = a(t) = 2t^{-3} + 6t^{-4} - 12t^{-5} + 20t^{-6}$

8. For $y = \frac{4}{9}x^3 z + 3x^2 z^7$, compute $\frac{dy}{dx}$ and $\frac{dy}{dz}$.

$y = \frac{4}{9}x^3 z + 3x^2 z^7$

For $\frac{dy}{dx}$, x is the variable
z is the constant

$\frac{dy}{dx} = \frac{4}{3}x^2 z + 6xz^7$

or

$\frac{dy}{dx} = \frac{4}{3}z \cdot x^2 + 6z^7 \cdot x$

For $\frac{dy}{dz}$, z is the variable
x is the constant

$\frac{dy}{dz} = \frac{4}{9}x^3 + 21x^2z^6$

9. Write the equation for the line tangent to the curve $y = x\sqrt{x} - \frac{9}{2x}$ at $x = 4$.

$y = x^{3/2} - \frac{9}{2}x^{-1}$

$\frac{dy}{dx} = \frac{3}{2}x^{1/2} + \frac{9}{2}x^{-2}$

$\left.\frac{dy}{dx}\right|_{x=4} = \frac{3}{2}\sqrt{4} + \frac{9}{2(4)^2}$

$= 3 + \frac{9}{32}$

$= \frac{105}{32}$

$y(4) = 4\sqrt{4} - \frac{9}{2(4)} = 8 - \frac{9}{8} = \frac{55}{8}$

Tangent Line: $y = f'(a)(x-a) + f(a)$

$y = \frac{105}{32}(x-4) + \frac{55}{8}$

10. Find the equation of the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point $(2, 1)$. What is the smallest slope on the curve?

The y-value of the vertex.

$y = x^3 - 4x + 1$

$\frac{dy}{dx} = m_{tan} = 3x^2 - 4$

$m_{tan}\big|_{x=2} = 3(2)^2 - 4$

$= 8$

$m_{\perp} = -\frac{1}{8}$

$y = -\frac{1}{8}(x-2) + 1$

$\frac{dy}{dx}$ is a quadratic function with a minimum of -4

so y has a smallest slope of $m = -4$.

11. Determine the values of a and b such that f is differentiable everywhere.

$$f(x) = \begin{cases} x^4 + bx + 2 & \text{for } x \le 0 \\ \frac{1}{2}e^x + a & \text{for } x > 0 \end{cases}$$

First $f(x)$ must be continuous.

1. $f(0) = 0^4 + b(0) + 2 = 2$
2. $\lim_{x \to 0^-} f(x) = 2$ and $\lim_{x \to 0^+} f(x) = \frac{1}{2}e^0 + a = \frac{1}{2} + a$

so $2 = \frac{1}{2} + a$

$\frac{3}{2} = a$

for $f(x)$ to be continuous at 0.

$$f'(x) = \begin{cases} 4x^3 + b & \text{for } x < 0 \\ \frac{1}{2}e^x & \text{for } x > 0 \end{cases}$$

Need $f'(x)$ to be continuous at 0.

$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} (4x^3 + b) = b$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{1}{2}e^x\right) = \frac{1}{2}$

so $f(0) = 1/2$

$b = 1/2$ for $f'(x)$ to be defined and continuous at $x = 0$

3.2 The Product and Quotient Rules

12. For $y = (1 + x - 4\sqrt{x})(e^x)$, find $\frac{dy}{dx}$.

$y = (1 + x - 4x^{1/2})(e^x)$

$\frac{dy}{dx} = (1 - 2x^{-1/2})e^x + (e^x)(1 + x - 4x^{1/2})$

$\frac{dy}{dx} = e^x(1 - 2x^{-1/2} + 1 + x - 4x^{1/2})$

$\frac{dy}{dx} = e^x(-2x^{-1/2} + 2 + x - 4x^{1/2})$

13. Compute the derivative of $f(x) = ax^{1/3}(x + 1) - x^{-3/4}$ with respect to x.

$f(x) = a \cdot x^{1/3}(x+1) - x^{-3/4}$

$\frac{df(x)}{dx} = \frac{1}{3}a \cdot x^{-2/3}(x+1) + (1)a \cdot x^{1/3} + \frac{3}{4}x^{-7/4}$

$= \frac{1}{3}ax^{1/3} + \frac{1}{3}ax^{-2/3} + ax^{1/3} + \frac{3}{4}x^{-7/4}$

$\frac{df(x)}{dx} = \frac{4}{3}ax^{1/3} + \frac{1}{3}ax^{-2/3} + \frac{3}{4}x^{-7/4}$

14. Suppose $f(2) = 3$ and $f'(2) = \frac{1}{4}$, determine $\frac{d}{dx}[xf(x)]$ at $x = 2$.

$$\frac{d}{dx}\left[x\, f(x)\right] = (1)\, f(x) + f'(x)\cdot x$$

$$\frac{d}{dx}\left[x f(x)\right]\Big|_{x=2} = f(2) + 2\cdot f'(2)$$

$$= 3 + 2\cdot\frac{1}{4}$$

$$= 3 + 1/2$$

$$= \frac{7}{2}$$

15. Compute $g'(p)$ for $g(p) = \frac{3e^p}{6p^2 - 8p}$.

$$g(p) = \frac{3e^p}{6p^2-8p}$$

$$g'(p) = \frac{3e^p(6p^2-8p) - (12p-8)(3e^p)}{(6p^2-8p)^2}$$

$$g'(p) = \frac{3e^p(6p^2-8p-12p+8)}{(6p^2-8p)^2}$$

$$g'(p) = \frac{3e^p(6p^2-20p+8)}{(6p^2-8p)^2}$$

16. Given $y = \frac{x^3 - 11}{1 - x^2}$, compute both the first and second derivative of y with respect to x.

$$y = \frac{x^3-11}{1-x^2}$$

$$\frac{dy}{dx} = \frac{3x^2(-x^2+1) - (-2x)(x^3-11)}{(1-x^2)^2}$$

$$\frac{dy}{dx} = \frac{-3x^4+3x^2+2x^4-22x}{(1-x^2)(1-x^2)}$$

$$\frac{dy}{dx} = \frac{-x^4+3x^2-22x}{x^4-2x^2+1}$$

$$\frac{d^2y}{dx^2} = \frac{(-4x^3+6x-22)(x^4-2x^2+1) - (4x^3-4x)(-x^4+3x^2-22x}{(x^4-2x^2+1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-2x^5+66x^4-4x^3-44x^2+6x-22}{(x^4-2x^2+1)^2}$$

17. Differentiate $f(x) = \dfrac{ax^2}{k^2 - x^2}$ with respect to x. Assume that a and k are positive constants. Identify the values of x for which $f(x)$ is not differentiable.

$f(x) = \dfrac{ax^2}{K^2 - x^2}$

Domain: $K^2 - x^2 \neq 0$
$x^2 \neq K^2$
$x \neq \pm K$

$f'(x) = \dfrac{2ax(K^2-x^2) - (-2x)ax^2}{(K^2-x^2)^2}$

$f'(x) = \dfrac{2axK^2 - 2ax^3 + 2ax^3}{(K^2-x^2)^2}$

$f'(x) = \dfrac{2aK^2 x}{(K^2-x^2)^2}$

Domain: $K^2 - x^2 \neq 0$
$x \neq \pm K$

f(x) is not differentiable when f(x) is undefined and when f'(x) is undefined

f(x) is not differentiable for $x \neq \pm K$

18. Write the equation of the line tangent to the curve $y = \dfrac{8}{x^2+4}$ at the point $(2, 1)$.

$y = \dfrac{8}{x^2+4}$

$y(2) = 1$

$y'(x) = \dfrac{0(x^2+4) - 2x(8)}{(x^2+4)^2}$

$y'(x) = \dfrac{-16x}{(x^2+4)^2}$

$y'(2) = \dfrac{-16(2)}{((2)^2+4)^2} = \dfrac{-32}{64} = -\dfrac{1}{2}$

Tangent Line: $y = f'(a)(x-a) + f(a)$

$y = \frac{-1}{2}(x - 2) + 1$

19. The functions f and g that satisfy the properties as shown in the table. Compute the indicated quantity.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	-3	3	5
1	$\boxed{2}$	$\boxed{9}$	7	11
2	-5	0	2	10
3	4	-1	$\boxed{-4}$	$\boxed{8}$

Compute the derivative first, then use the table.

(a) $H'(3)$ if $H(x) = (x^3 + 2)g(x)$

$H(x) = (x^3+2)g(x)$

$H'(x) = 3x^2 \cdot g(x) + g'(x)(x^3+2)$

$H'(3) = 3(3)^2 g(3) + g'(3)((3)^3+2)$

$= 27 \cdot (-4) + 8\,(29)$

$= -108 + 232$

$H'(3) = 124$

(b) $\frac{d}{dx}\left(\frac{x^3}{f(x)}\right)\Big|_{x=1}$

$$\frac{d}{dx}\left(\frac{x^3}{f(x)}\right) = \frac{3x^2 \cdot f(x) - f'(x) \cdot x^3}{(f(x))^2}$$

$$\frac{d}{dx}\left(\frac{x^3}{f(x)}\right)\Big|_{x=1} = \frac{3(1)^2 \cdot f(1) - f'(1) \cdot (1)^3}{(f(1))^2}$$

$$= \frac{3 \cdot (2) - 9\,(1)}{(2)^2}$$

$$= \frac{6-9}{4}$$

$$= \frac{-3}{4}$$