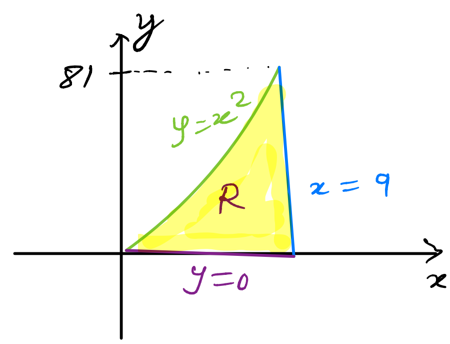
Example 1 (15.2)**.** *Set up double integrals with both orders of integration for* $\iint_R f(x,y)\,dA$, *where* R *is the plane region bounded by* $y = x^2$, $x = 9$ *and* $y = 0$.

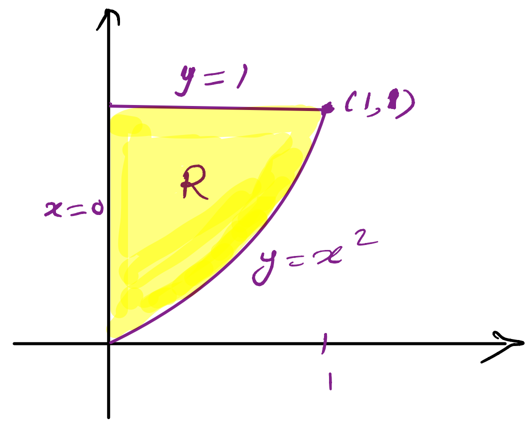
Type I: $\iint_R f(x,y)\,dA = \int_0^9 \int_0^{x^2} f(x,y)\,dy\,dx$

Type II: $\iint_R f(x,y)\,dA = \int_0^{81} \int_{\sqrt{y}}^{9} f(x,y)\,dx\,dy$



Example 2 (15.2). *Evaluate the integral* $\int_0^1 \int_{x^2}^1 \sqrt{y}\, e^{y^2}\, dy\, dx$ *by reversing the order of integration.*

$$\int_0^1 \int_{x^2}^1 \sqrt{y}\, e^{y^2}\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} \sqrt{y}\, e^{y^2}\, dx\, dy$$



$$= \int_0^1 \sqrt{y}\, e^{y^2} \cdot \sqrt{y}\, dy$$

$$= \int_0^1 y\, e^{y^2}\, dy$$

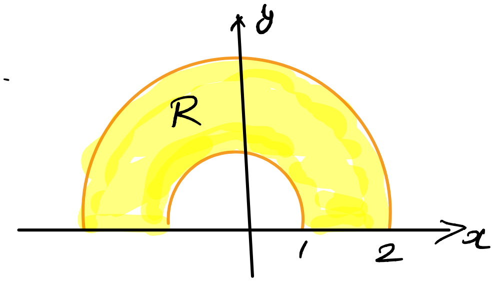
sub. $u = y^2 \Rightarrow du = 2y\, dy$

$$= \frac{1}{2} \int e^u\, du = \frac{1}{2} e^u \Big|$$

$$= \frac{1}{2} e^{y^2} \Big|_{y=0}^{1} = \frac{1}{2}\left[e - 1\right]$$

Example 3 (15.3)**.** *(a) Set up (but do NOT evaluate) a double integral using polar coordinates to compute the volume of the solid E that lies below the paraboloid $z = 1 + x^2 + y^2$ and above the region R between the circles $x^2+y^2 = 1$ and $x^2+y^2 = 4$ in the half-space $y \geq 0$.*

$R = \{(r, \theta) : 1 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}$.



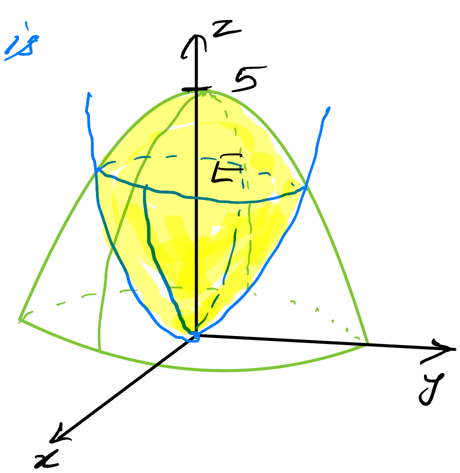
Volume $V(E) = \iint_R (1+x^2+y^2)\, dA$

$= \int_0^{\pi}\int_1^2 (1+r^2)\cdot r\, dr\, d\theta$

Easy to compute.

(b) Use polar coordinates to compute the volume of the solid E that lies below the paraboloids $z = 5 - x^2 - y^2$ and $z = 4x^2 + 4y^2$.

The intersection of $z = 5-x^2-y^2$ and $z = 4x^2+4y^2$ is
$5-x^2-y^2 = 4x^2+4y^2 \Rightarrow \boxed{x^2+y^2=1}$



Projection of E onto xy plane is
$D: x^2+y^2 \leq 1.$

That is, $D = \{(r,\theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\}$.

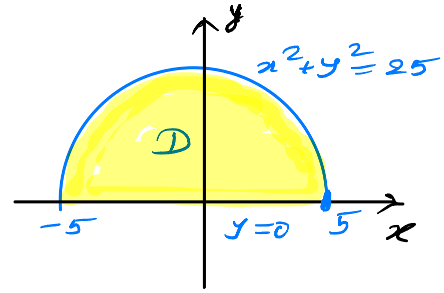
$V(E) = \iint_D (\text{top} - \text{bottom})\, dA$

$= \iint_D [(5-x^2-y^2) - (4x^2+4y^2)\, dA = \iint_D [5-5(x^2+y^2)]\, dA$

$= \int_0^{2\pi}\int_0^1 [5-5r^2]\cdot r\, dr\, d\theta$

Example 4 (15.3). *Rewrite (but do NOT evaluate) the integrals using polar coordinates.*

(a) $\int_{-5}^{5}\int_{0}^{\sqrt{25-x^2}} e^{x^2+y^2}\,dy\,dx.$ $= I$



Region of integration:

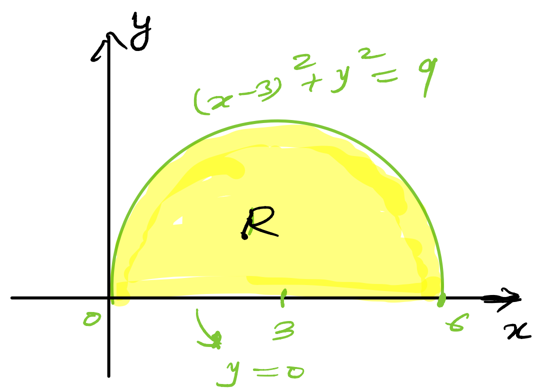
$D = \{(x,y) : -5 \le x \le 5,\ 0 \le y \le \sqrt{25-x^2}\}$

$y = \sqrt{25-x^2} \Rightarrow x^2+y^2 = 25.$

$D = \{(r,\theta) : 0 \le r \le 5,\ 0 \le \theta \le \pi\}.$

So, $I = \int_0^{\pi}\int_0^{5} e^{r^2}\cdot r\,dr\,d\theta$

Easy to compute.



(b) $\int_{0}^{6}\int_{0}^{\sqrt{6x-x^2}} (x^2+y^2)\,dy\,dx$ $= I$

$0 \le x \le 6,\ 0 \le y \le \sqrt{6x-x^2}.$

$y = \sqrt{6x-x^2} \Rightarrow \boxed{x^2+y^2 = 6x}$

Completing the square,

$x^2-6x+9+y^2 = 9$

$\boxed{(x-3)^2+y^2 = 9}$

$r^2 = 6r\cos\theta \Rightarrow r=0,\ r = 6\cos\theta.$

Since both $x \ge 0, y \ge 0$, $0 \le \theta \le \pi/2.$

So, $R = \{(r,\theta) : 0 \le r \le 6\cos\theta,\ 0 \le \theta \le \pi/2\}$

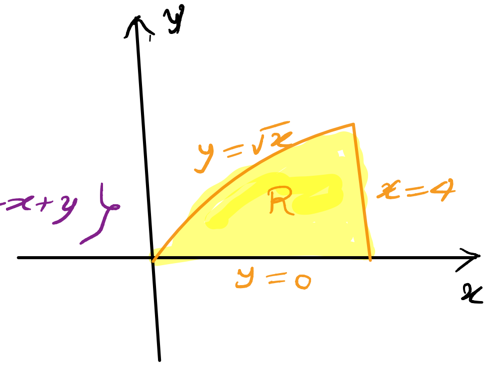
$I = \int_0^{\pi/2}\int_0^{6\cos\theta} r^2\cdot r\,dr\,d\theta$

Easy to calculate.

Example 5 (15.6). *Evaluate* $\iiint_E 6x\,dV$, *where* E *lies under the plane* $x+y-z+2=0$ *and above the region in the xy-plane bounded by the curves* $y=\sqrt{x}$, $y=0$ *and* $x=4$.

$x+y-z+2=0 \Rightarrow z = 2+x+y$

So, $\boxed{0 \le z \le 2+x+y}$



$E = \{(x,y,z) : 0 \le x \le 4,\ 0 \le y \le \sqrt{x},\ 0 \le z \le 2+x+y\}$

$$\iiint_E 6x\,dV = \int_0^4 \int_0^{\sqrt{x}} \int_0^{2+x+y} 6x\,dz\,dy\,dx$$

$2x + x^2 + xy$

$$= 6\int_0^4 \int_0^{\sqrt{x}} x(2+x+y)\,dy\,dx$$

$$= 6\int_0^4 2xy + x^2y + \frac{xy^2}{2} \Big|_{y=0}^{\sqrt{x}} dx$$

$$= 6\int_0^4 2x^{3/2} + x^{5/2} + \frac{x^2}{2}\,dx$$

$2^{\cancel{2}\cdot 5/\cancel{2}}$

$2^{\cancel{2}\cdot 7/\cancel{2}}$

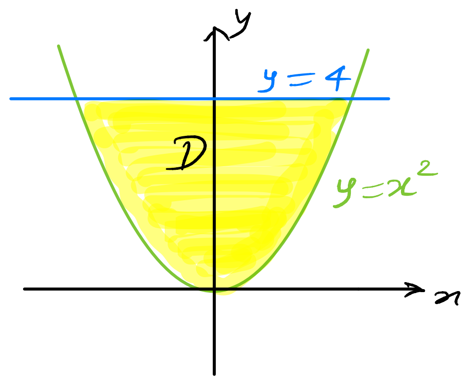
$$= 6\left[2\cdot\frac{2}{5}\cdot x^{5/2} + \frac{2}{7}x^{7/2} + \frac{x^3}{6}\right]_{x=0}^{4}$$

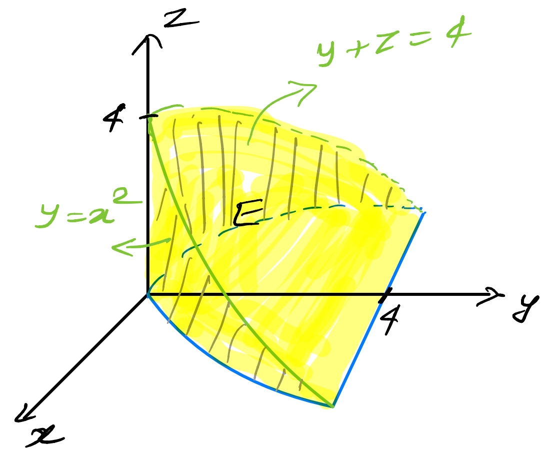
4^3

$$= 6\left[\frac{128}{5} + \frac{256}{7} + \frac{64}{6}\right]$$

Example 6 (15.6). *Set up the triple integral for the volume of the solid E in the order $dz\,dx\,dy$, where E is the solid bounded by $y = x^2$, $z = 0$, and $y + z = 4$.*

$y = x^2 \Rightarrow x = \pm\sqrt{y}$



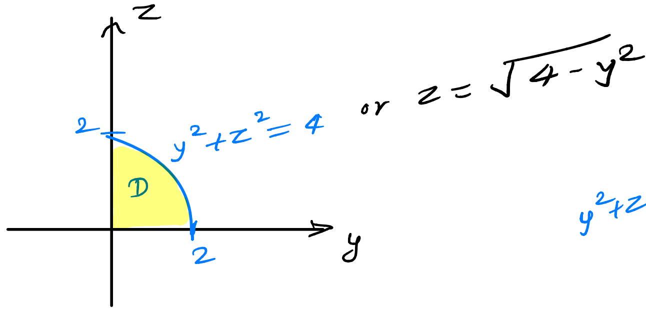


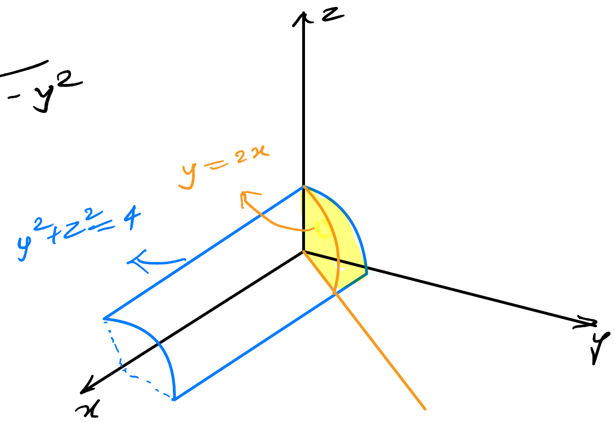
$$E = \{(x,y,z) : 0 \le y \le 4,\ -\sqrt{y} \le x \le \sqrt{y},\ 0 \le z \le 4-y\}$$

$$V(E) = \iiint_E dV$$

$$= \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{4-y} dz\,dx\,dy$$

Example 7 (15.6). *Set up the triple integral in the order* <u>$dx\,dz\,dy$</u> *for* $\displaystyle\iiint_E z\,dV$, *where* E *is the solid bounded by* $y^2+z^2=4$, $x=0$, $y=2x$ *and* $z=0$ *in the first octant.*

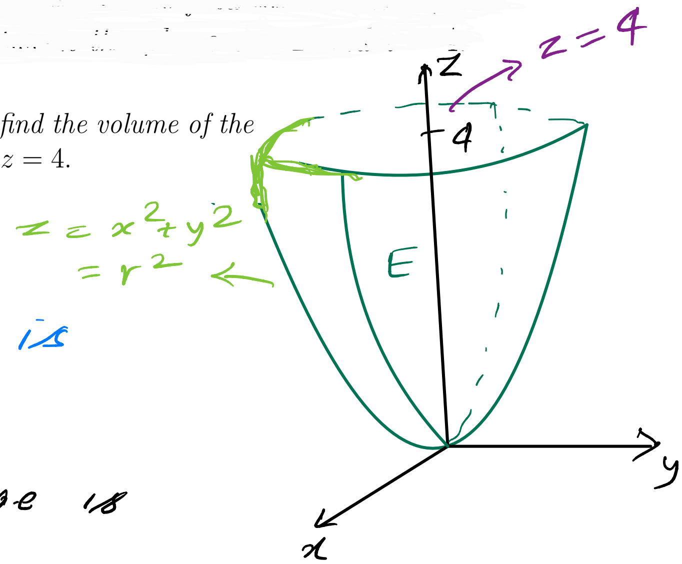




$$E=\{(x,y,z):0\le y\le 2,\ 0\le z\le\sqrt{4-y^2},\ 0\le x\le \tfrac{y}{2}\}$$

$$\iiint_E z\,dV=\int_0^2\int_0^{\sqrt{4-y^2}}\int_0^{y/2} z\,dx\,dz\,dy$$

Example 8 (15.7). *Set up an integral using cylindrical coordinates to find the volume of the solid E that lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 4$.*



Intersection of $z = x^2+y^2$ and $z=4$ is
$x^2+y^2=4$

So, the projection of E onto xy plane is

$R: \quad x^2+y^2 \le 4.$

That is $R = \{(r,\theta): 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$.

And $E = \{(r,\theta,z): 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ r^2 \le z \le 4\}$.

$$V(E) = \iiint_E dV$$

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^4 r\,dz\,dr\,d\theta$$

(Easy to compute).

Example 9 (15.7). *Use cylindrical coordinates to set up a triple integral for the volume of the solid E*

(a) that lies between the paraboloids $z = 8 - x^2 - y^2$ and $z = 3x^2 + 3y^2$.

$z = 8 - r^2 \qquad z = 3r^2$

Intersection of the paraboloids : $8 - x^2 - y^2 = 3x^2 + 3y^2$

$\Rightarrow \boxed{x^2 + y^2 = 2}$

$E = \{(r, \theta, z) : 0 \le r \le \sqrt{2},\ 0 \le \theta \le 2\pi,\ 3r^2 \le z \le 8 - r^2\}$.

$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{3r^2}^{8-r^2} r \, dz \, dr \, d\theta$$

(b) that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

$z^2 = 4 - r^2 \Rightarrow z = \pm\sqrt{4 - r^2}$

$0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{1} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$$

(c) *that lies below the paraboloid* $z = 24 - x^2 - y^2$ *and above the cone* $z = 2\sqrt{x^2 + y^2}$.

$z = 24 - r^2$ $\qquad z = 2r$

Intersection of the surfaces:

$$24 - r^2 = 2r$$

$$\Rightarrow r^2 + 2r - 24 = 0$$

$$\Rightarrow (r+6)(r-4) = 0 \Rightarrow r \cancel{=} -6 \text{ or } \boxed{r = 4}$$

$E = \{(r, \theta, z) : 0 \le r \le 4,\ 0 \le \theta \le 2\pi,\ 2r \le z \le 24 - r^2\}$.

$$V(E) = \int_0^{2\pi} \int_0^4 \int_{2r}^{24-r^2} r \, dz \, dr \, d\theta.$$

Example 10 (15.8). *Evaluate* $\displaystyle\iiint_E \frac{x}{1+(x^2+y^2+z^2)^2}\,dV$, *where* E *is the solid between the spheres* $\underbrace{x^2+y^2+z^2=1}_{\rho=1}$ *and* $\underbrace{x^2+y^2+z^2=4}_{\rho=2}$.

$\rightsquigarrow \boxed{1 \le \rho \le 2}$

Clearly, $\boxed{0 \le \phi \le \pi}$ and $\boxed{0 \le \theta \le 2\pi}$.

$E = \{(\rho, \theta, \phi) : 1 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$.

$$\iiint_E \frac{x}{1+(x^2+y^2+z^2)^2}\,dV = \int_0^{\pi}\int_0^{2\pi}\int_1^2 \frac{\rho\sin\phi\cos\theta}{1+\rho^4}\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \left(\int_0^{\pi}\sin^2\phi\,d\phi\right)\left(\int_0^{2\pi}\cos\theta\,d\theta\right)\left(\int_1^2\frac{\rho^3}{1+\rho^4}\,d\rho\right)$$

$\int_0^{2\pi}\cos\theta\,d\theta \to 0$

$= 0$

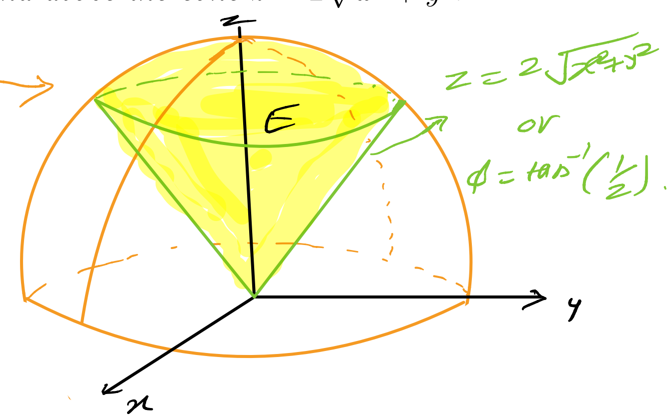
Example 11 (15.8)**.** *Use spherical coordinates to set up a triple integral for the volume of the solid E that lies below the sphere $x^2 + y^2 + z^2 = 5$ and above the cone $z = 2\sqrt{x^2 + y^2}$.*

$$z = 2\sqrt{x^2+y^2}$$

$$\Downarrow$$

$$\rho\cos\phi = 2r = 2\rho\sin\phi$$

$$\Rightarrow \frac{1}{2} = \tan\phi \Rightarrow \phi = \tan^{-1}\left(\tfrac{1}{2}\right)$$



The intersection of $x^2+y^2+z^2=5$ and $z = 2\sqrt{x^2+y^2}$ is

$$x^2+y^2+[4(x^2+y^2)] = 5$$

$$\Rightarrow \boxed{x^2+y^2=1}$$

So, $\boxed{0 \le \theta \le 2\pi}$, $\boxed{0 \le \phi \le \tan^{-1}(\frac{1}{2})}$, $\boxed{0 \le \rho \le \sqrt{5}}$,

$$V(E) = \iiint_E dV = \int_0^{\tan^{-1}(\frac{1}{2})}\int_0^{2\pi}\int_0^{\sqrt{5}} \rho^2 \sin\phi \cdot d\rho\, d\theta\, d\phi$$

Example 12 (15.8)**.** *Convert (but do NOT evaluate) the integral*

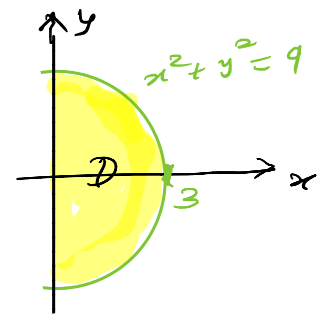
$$I = \int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$

into an iterated integral with spherical coordinates.

$z = \pm\sqrt{9-x^2-y^2} \Rightarrow \boxed{x^2+y^2+z^2=9} \Rightarrow \rho = 3.$

Note that $x \geq 0$.

$$\boxed{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}}$$



Clearly, $0 \leq \rho \leq 3$, $0 \leq \phi \leq \pi$.

$$I = \int_0^{\pi}\int_{-\pi/2}^{\pi/2}\int_0^3 (\rho^2 \sin^2\phi \cos^2\theta)\cdot \rho \cdot \overbrace{\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi}^{dV}$$

Example 13 (15.8). *Consider the density function $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ on the solid $E:\ x^2 + y^2 + z^2 \le 4,\ z \ge 0$. Find the mass.*

$E = \{(\rho, \theta, \phi) : 0 \le \rho \le 2, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}\}$.

Mass $m = \iiint_E \rho(x,y,z)\, dV$

$= \int_0^{\pi/2} \int_0^{2\pi} \int_0^2 \rho \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

$= \left(\int_0^{\pi/2} \sin\phi \, d\phi\right) \left(\int_0^{2\pi} d\theta\right) \left(\int_0^2 \rho^3 \, d\rho\right)$

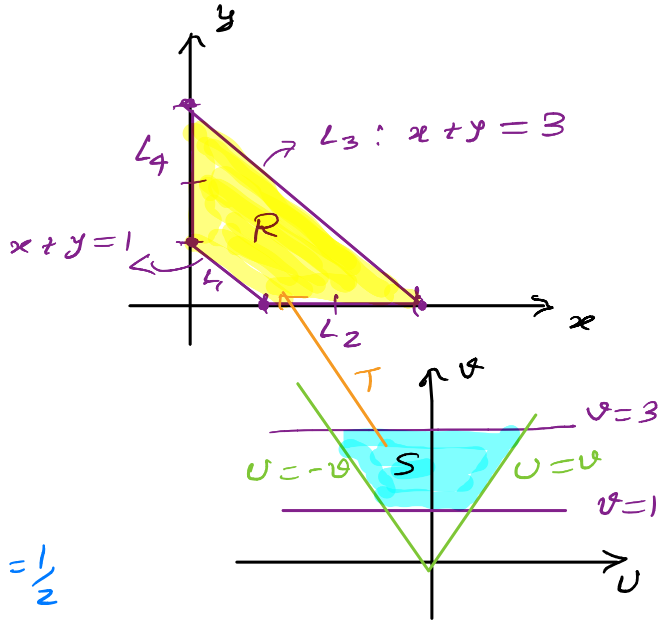
$= (1)\ (2\pi) \cdot (4)$

$= 8\pi$

Example 14 (15.9)**.** *Use the transformation $u = x-y$ and $v = x+y$ to evaluate* $\displaystyle\iint\limits_R \frac{x-y}{x+y}\,dA$, *where R is the trapezoidal region with vertices* $(1,0),(3,0),(0,3),$ *and* $(0,1)$.

$u + v = 2x \Rightarrow \boxed{x = \frac{1}{2}(u+v)}$

$v - u = 2y \Rightarrow \boxed{y = \frac{1}{2}(v-u)}$



Jacobian: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Transformation of L_1: $x+y=1 \Rightarrow \frac{1}{2}(u+v) + \frac{1}{2}(v-u) = 1 \Rightarrow \boxed{v=1}$

" " L_2: $x = 0 \Rightarrow \frac{1}{2}(u+v) = 0 \Rightarrow \boxed{u = -v}$

" " L_3: $x+y = 3 \Rightarrow \frac{1}{2}(u+v) + \frac{1}{2}(v-u) = 3 \Rightarrow \boxed{v = 3}$

" " L_4: $y = 0 \Rightarrow \frac{1}{2}(v-u) = 0 \Rightarrow \boxed{u = v}$

$$\iint\limits_R \frac{x-y}{x+y}\,dA = \iint\limits_S \frac{u}{v}\cdot\left|\frac{1}{2}\right| dA'$$

$dA' \to$ dudv or dvdu

$$= \frac{1}{2}\int_1^3\int_{-v}^{v} \frac{u}{v}\,du\,dv = \frac{1}{2}\int_1^3 \frac{1}{v}\cdot\frac{u^2}{2}\Big|_{-v}^{v}\,dv$$

$$= \frac{1}{4}\int_1^3 (0)\,dv = 0.$$

Example 15 (15.9). *Evaluate the integral* $\displaystyle\iint_R (x-2y)\,e^{3x-y}\,dA$ *by using an appropriate change of variables, where R is the region bounded by the lines*

$$\underbrace{x-2y=0}_{L_1},\ \underbrace{x-2y=4}_{L_2},\ \underbrace{3x-y=1}_{L_3},\ \text{and } \underbrace{3x-y=8}_{L_4}.$$

Consider the change of variables : $u = x-2y$, $v = 3x-y$. Then

$$u - 2v = x - \cancel{2y} - 6x + \cancel{2y} \Rightarrow \boxed{x = \tfrac{1}{5}(2v-u)}$$

$$3u - v = \cancel{3x} - 6y - \cancel{3x} + y \Rightarrow \boxed{y = \tfrac{1}{5}(v-3u)}$$

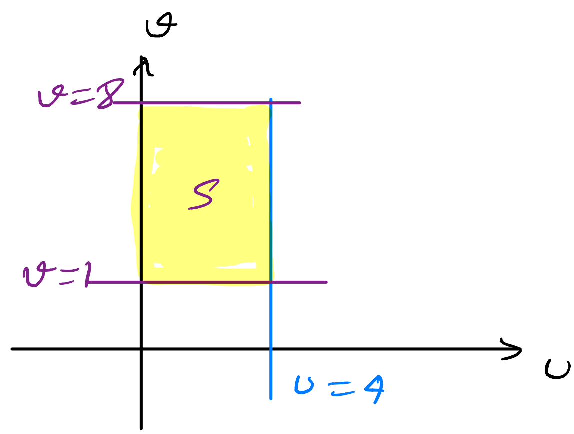
Jacobian : $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix} = -\frac{1}{25} + \frac{6}{25} = \frac{1}{5}$

L_1 : $x-2y = 0 \Rightarrow u = 0$

L_2 : $x-2y = 4 \Rightarrow u = 4$

L_3 : $3x-y = 1 \Rightarrow v = 1$

L_4 : $3x-y = 8 \Rightarrow v = 8$



$$\iint_R (x-2y)\,e^{3x-y}\,dA = \iint_S u\,e^{v}\left|\tfrac{1}{5}\right| dA', \quad dA' \to du\,dv \text{ or } dv\,du$$

$$= \frac{1}{5}\int_1^8\int_0^4 u\,e^{v}\,du\,dv = \frac{1}{5}\left(\int_1^8 e^{v}\,dv\right)\left(\int_0^4 u\,du\right)$$

$$= \frac{1}{5}\left(e^8 - e\right)\cdot 8 = \frac{8}{5}\left(e^8 - e\right)$$