Math 308: Week-in-Review 8 (6.3 - 6.5)

6.1-6.6 Laplace Transform

Review

- Definition of the Laplace transform

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\, dt$$

- General strategy for solving differential equations with the Laplace transform
    1. Laplace transform
    2. Solve for $Y(s)$
    3. Inverse transform

- Common Laplace transforms

$f(t)$	$F(s)$	defined for
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n (n = 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$e^{at} t^n (n = 1, 2, \ldots)$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at} \sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$u_c(t) (c \geq 0)$	$\frac{e^{-cs}}{s}$	$s > 0$
$\delta(t - c) (c \geq 0)$	e^{-cs}	

- Shift theorems

$$\mathcal{L}\{u_c(t) f(t - c)\} = e^{-cs} F(s)$$
$$\mathcal{L}\{u_c(t) f(t)\} = e^{-cs} \mathcal{L}\{f(t + c)\}$$
$$\mathcal{L}^{-1}\{e^{-cs} F(s)\} = u_c(t) f(t - c)$$
$$\mathcal{L}^{-1}\{F(s - c)\} = e^{ct} f(t)$$

1 6.2: Solving ODEs with Laplace Transforms

Review

- Laplace transform of derivatives
- $\mathcal{L}\{f'\} =$
- $\mathcal{L}\{f''\} =$
- $\mathcal{L}\{f'''\} =$
- How to solve differential equations with the Laplace transform
  - Laplace transform
  - Solve for $Y(s)$
  - Inverse transform

1. Use the Laplace transform to solve the initial value problem

$$y'' - 3y' + 2y = 0, \quad y(0) = -2, \quad y'(0) = 1.$$

2. Use the Laplace transform to solve the initial value problem

$$y'' + 2y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

3. Use the Laplace transform to solve the initial value problem

$$y'' + 6y' + 9y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = 0.$$

4. Use the Laplace transform to solve the initial value problem

$$y''' - y' = 0, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = -1.$$

2 6.3: Step Functions

Review

- The unit step function $u_c(t)$ is defined by

- It can be used to write discontinuous functions into a single equation.
- The Laplace transform of $u_c(t)$ is

- Laplace transforms of shifts

- Inverse Laplace transform of shifts

5. Convert the following function to a piecewise function. Also, graph the function. Compute its Laplace transform.

$$f(t) = u_3(t) - 2u_5(t)$$

6. Convert the following function to a piecewise function. Compute its Laplace transform.

$$f(t) = t - \cos(t-2)u_2(t) - tu_3(t)$$

7. Convert the following piecewise function into a form that involves step functions.

$$g(t) = \begin{cases} 0, & t < 2 \\ 3, & 2 \leq t < 5 \\ \sin(3t), & t \geq 5 \end{cases}$$

8. Convert the following piecewise function into a form that involves step functions. Compute its Laplace transform.

$$g(t) = \begin{cases} 3t, & t < 5 \\ e^{3t}, & t \geq 5 \end{cases}$$

9. Solve the initial value problem.

$$f'' + 4f = u_3(t), \quad f(0) = 0, \quad f'(0) = 0.$$

10. Solve the initial value problem.

$$w'' + 2w' = \begin{cases} 3, & t < 5 \\ 0, & t \geq 5 \end{cases}, \quad w(0) = 0, \quad w'(0) = 0.$$

11. Consider a spring and mass system with a 5 kg mass hanging on a spring. When the mass is hung on the spring, the spring extends 50 cm. The mass experiences a damping force of 8 N when the mass is moving 2 m/s. The mass starts from equilibrium at rest, but there is an external force $\cos(t)$ that lasts for the first 3π seconds. Write down the initial value problem that describes this situation.

3 6.6: Delta Functions

Review

- The Dirac delta function $\delta(t-c)$ is defined by
- It can be used to model instantaneous impulses or point sources in differential equations.
- The Laplace transform of $\delta(t-c)$ is
- Laplace transforms involving delta functions
- Inverse Laplace transform involving delta functions

12. Find the Laplace transform of the following function:

$$f(t) = t^2\delta(t-3) + e^t\delta(t-5).$$

13. Solve the initial value problem:

$$y'' + 2y' + y = \delta(t-3), \quad y(0) = 1, \quad y'(0) = -1.$$

14. A 2 kg mass is suspended from a spring and damper. When the mass is hung at rest, it stretches the spring by 2 meters. When the mass moves at 1 m/s, the damper exerts a resistive force of 4 N. At $t = 2$ seconds, the system is struck with a hammer, delivering an instantaneous impulse force of magnitude 3N. The mass starts motion from equilibrium with an initial upward velocity of 0.5 m/s.

    (a) Determine the spring constant k and damping coefficient c.

    (b) Write the governing differential equation for the displacement $u(t)$.

    (c) Solve for $u(t)$ and describe the motion of the system.

    (Use $g = 10\,\text{m/s}^2$.)