Math 150 - Week-In-Review 11

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Exam 3 review

Summary of the Topics:

a) **Chapter 6**

6.1 & 6.2 Solving system of linear or nonlinear equations.

b) **Chapter 7**

7.1 **Degree and Radian Measures**.

acute vs. obtuse angles, supplementary angle, complementary angle, coterminal angle, convert from radian to degree and vice versa, arc length, linear and angular velocity.

7.2 **Sine and cosine functions**

Their relation with right triangle, Pythagrean theorem, reference angles, how to evaluate sine and cosine of a given angle (on a unit circle or any other circle with radius r)

7.3 **Graphs of sine and cosine**

properties of sine and cosine function, amplitude, period, domain, range, even/odd, sketch

7.4 **Other trig functions**

Their relation with right triangle, how to compute trig functions of a given angle (on a unit circle or any other circle with radius r)

7.5 **Graphs of tangent, cotangent, secant and cosecand**

All their properties such as period, domain, range, even/odd, how to sketch their transformations, how to find their vertical asymptotes.

7.6 **Inverse trig functions**

All their properties such as domain, range, compute compositions, their graph, their asymptotes (if any), compute exact value of inverse trig function by giving an answer in its range.

c) **Chapter 8**

8.1 **Fundamental and Pythagrean identities**

Need to memorize all IDs in this section. How to use them to find trig equations, how to verify IDs.

8.2 **Other trig identities**

No need to memorize

Need to memorize: Even/odd IDs, sum and difference IDs for sine, cosine and ~~tangent~~, Double angle IDs, Power reduction IDs. How to use these IDs to compute exact value of your trig functions, as well as using the rest of the IDs if given to you.

8.3 **Solving equations involving trig functions**

How to solve trig equations. First finding solutions using inverse notation, then finding exact values like you did in 7.6 (if known angle)

d) **Python**

For more practice, you can review the problems in Week-in-Review 8, 9 and 10 as well.

1. Find all solutions to the equation $\dfrac{\cos(2x)}{\cos^2 x} = 1$.

Note: $\cos(x) \neq 0$

$x \neq \frac{\pi}{2} + k\pi$

$$\frac{\cos^2(x) - \sin^2(x)}{\cos^2(x)} = 1$$

$1 - \tan^2(x) = 1 \implies \tan^2(x) = 0$

$\tan(x) = 0$

$x = + k\pi$ where k is any integer

2. Solve the equation $5\sin(\theta)\cot(\theta) + 4\cot(\theta) = 0$

Note: $\sin\theta \neq 0$

$\theta \neq k\pi$

$\cot(\theta)\left(5\sin\theta + 4\right) = 0$

$\cot\theta = 0$ & $\sin\theta = -\frac{4}{5}$

$\boxed{\theta_3 = \frac{\pi}{2} + k\pi}$

$\boxed{\theta_1 = \arcsin\left(\frac{-4}{5}\right) + 2k\pi}$ in Q IV

& $\boxed{\theta_2 = \pi - \arcsin\left(\frac{-4}{5}\right) + 2k\pi}$ in Q III

3. Find all solutions for $\tan(2x) + \tan x = 0$ on $[0, 2\pi)$

Note: $\cos(2x) \neq 0$

$2x \neq \frac{\pi}{2} + k\pi$

$x \neq \frac{\pi}{4} + \frac{k\pi}{2}$

$$\frac{2\tan(x)}{1-\tan^2 x} + \tan(x) = 0$$

$$\frac{2\tan(x) + \tan(x) - \tan^3(x)}{1-\tan^2(x)} = 0$$

$1 - \tan^2(x) \neq 0$

$\tan(x) \neq \pm 1$

$x \neq \frac{\pi}{4} + k\pi$

$x \neq -\frac{\pi}{4} + k\pi$

$$3\tan(x) - \tan^3(x) = 0$$

$$\tan(x)\left(3 - \tan^2(x)\right) = 0$$

$\tan(x) = 0$ $\quad x_1 = k\pi$

$\tan(x) = +\sqrt{3}$

$x_2 = \arctan(\sqrt{3}) + k\pi$

$\tan(x) = -\sqrt{3}$

$x_3 = \arctan(-\sqrt{3}) + k\pi$

$x = k\pi, \quad \frac{\pi}{3} + k\pi, \quad -\frac{\pi}{3} + k\pi$

Solutions on $[0, 2\pi)$: $\underbrace{0, \frac{\pi}{3}}_{k=0}, \underbrace{\pi, \frac{4\pi}{3}, \frac{2\pi}{3}}_{k=1}, \underbrace{\frac{5\pi}{3}}_{k=2}$

4. Verify the following identities. $\cot^2(t)$

a. $\dfrac{3\cot^3 t}{\csc t} = 3\cos t(\csc^2 t - 1)$

LHS: $\dfrac{3\cot(t)\cdot\cot^2(t)}{\csc(t)} = \dfrac{3\,\dfrac{\cos(t)}{\sin(t)}\cdot\cot^2(t)}{\dfrac{1}{\sin(t)}} = 3\,\dfrac{\cos t}{\cancel{\sin t}}\cdot\dfrac{\cancel{\sin t}}{1}\cdot\cot^2 t$

$= 3\cos(t)\cot^2(t)$

$= 3\cos(t)\left(\csc^2(t) - 1\right)$

b. $\tan x - \cot x = \sec x(2\sin x - \csc x)$

$\dfrac{\sin x}{\cos x} - \dfrac{\cos x}{\sin x} = \dfrac{1}{\cos x}\left(\sin x - \dfrac{\cos^2 x}{\sin x}\right) = \dfrac{1}{\cos x}\left[\sin x - \dfrac{(1-\sin^2 x)}{\sin x}\right]$

$= \dfrac{1}{\cos x}\left[\sin x - \dfrac{1}{\sin x} + \dfrac{\sin^2 x}{\sin x}\right] = \sec(x)\left[2\sin x - \csc x\right]$

5. Simplify the expression $\dfrac{(\sin(x)+\tan(x))^2 + \cos^2(x) - \sec^2(x)}{\tan(x)}$.

$= \dfrac{\sin^2(x) + \tan^2(x) + 2\sin(x)\tan x + \cos^2(x) - \sec^2(x)}{\tan(x)}$

$= \dfrac{1 - 1 + 2\sin x\tan x}{\tan x} = \dfrac{2\sin x\tan x}{\tan x} = \dfrac{2\sin x\,\cancel{\tan x}}{\cancel{\tan x}}$

$= 2\sin x$

6. Rewrite $\sin(x)\cos(3x) + \sin(3x)\cos(x)$ as a single expression.

$$= \sin(x + 3x) = \sin(4x)$$

7. Find the exact value for $\sin(2\theta)$, $\cos(2\theta)$, and $\cot(2\theta) - \csc(2\theta)$, if $\cos(\theta) = -\frac{6}{11}$ and θ is in QIII.

$$\boxed{\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(-\frac{6}{11}\right)^2 - 1 = 2\left(\frac{36}{121}\right) - 1 = \frac{72-121}{121} = -\frac{49}{121}}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2(\sin\theta)\left(-\frac{6}{11}\right)$$

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2\theta = 1 - \cos^2\theta$$

$$\sin^2\theta = 1 - \left(\frac{36}{121}\right) = \frac{121-36}{121} = \frac{85}{121}$$

QIII

$$\Rightarrow \sin\theta = -\sqrt{\frac{85}{121}} = -\frac{\sqrt{85}}{11}$$

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(-\frac{\sqrt{85}}{11}\right)\left(-\frac{6}{11}\right) = +\frac{12\sqrt{85}}{121}}$$

$$\cot(2\theta) = \frac{\cos(2\theta)}{\sin(2\theta)} = \frac{-\frac{49}{\cancel{121}}}{\frac{12\sqrt{85}}{\cancel{121}}} = -\frac{49}{12\sqrt{85}}$$

$$\csc(2\theta) = \frac{1}{\sin(2\theta)} = \frac{121}{12\sqrt{85}}$$

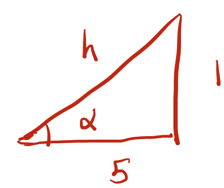
$$\Rightarrow \boxed{\cot(2\theta) - \csc(2\theta) = -\frac{49}{12\sqrt{85}} - \frac{121}{12\sqrt{85}} = \frac{-170}{12\sqrt{85}}}$$

8. Determine the exact value of x given $\arcsin(x) = 2\arctan\left(\frac{1}{5}\right)$.

$$\arcsin(x) = 2\underbrace{\arctan\left(\frac{1}{5}\right)}_{\alpha}$$

$$\arcsin x = 2\alpha \quad \overset{\text{equivalent to}}{\Longleftrightarrow} \quad x = \sin(2\alpha) \underset{\text{ID}}{\overset{\text{double angle}}{=}} 2\sin(\alpha)\cos(\alpha)$$

$$\alpha = \arctan\left(\frac{1}{5}\right) \Longleftrightarrow \tan(\alpha) = \frac{1}{5} \qquad \frac{\text{opp}}{\text{adj}}$$



$$h^2 = 5^2 + 1^2$$

$$h^2 = 26$$

$$h = \sqrt{26}$$

$$\sin\alpha = \frac{\text{opp.}}{\text{hyp.}} = \frac{1}{\sqrt{26}}$$

$$\cos\alpha = \frac{\text{adj}}{\text{hyp.}} = \frac{5}{\sqrt{26}}$$

$$\boxed{x = 2\sin(\alpha)\cos(\alpha) = 2\left(\frac{1}{\sqrt{26}}\right)\left(\frac{5}{\sqrt{26}}\right) = \frac{10}{26}}$$

9. Simplify the trigonometric expression.

$$\frac{\sec^2(x) - 1}{\sin^2(x)}$$

$$\frac{\sec^2(x) - 1}{\sin^2(x)} = \frac{\frac{1}{\cos^2 x} - 1}{\sin^2 x} = \frac{1 - \cos^2 x}{\cos^2 x \ \sin^2 x} = \frac{\sin^2 x}{\sin^2 x \cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

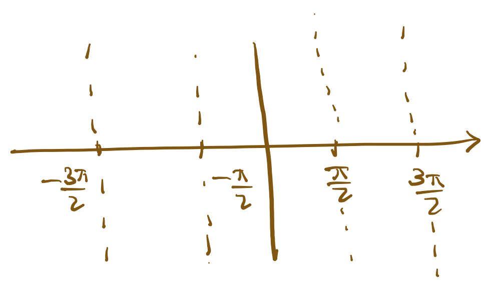
Or

$$\frac{\sec^2(x) - 1}{\sin^2(x)} = \frac{\tan^2(x)}{\sin^2(x)} = \frac{\frac{\cancel{\sin^2}(x)}{\cos^2(x)}}{\underbrace{\cancel{\sin^2}(x)}_{1}} = \frac{1}{\cos^2(x)} = \cos^2(x)$$

10. List all the vertical asymptotes of $h(x) = \sec(x)$ on the interval $[-2\pi, 2\pi]$.

$\sec(x) = \frac{1}{\cos(x)}$ Vertical asy at $x = \frac{\pi}{2} + k\pi$

on $\left[-2\pi, 2\pi\right]$: $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$



11. Simplify each composition, if possible.

$\cot\left[\arctan(-\sqrt{3})\right] = \underline{\cot\left(-\frac{\pi}{3}\right)} \overset{\text{odd}}{=} -\cot\left(\frac{\pi}{3}\right) = -\frac{1}{\sqrt{3}} \overset{\text{or}}{=} -\frac{\sqrt{3}}{3}$

$\sec\left[\arccos(\underbrace{-\sqrt{3}}_{<-1})\right] =$ DNE

$\arcsin\left[\sin\left(\frac{5\pi}{4}\right)\right] = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

$\arcsin\left[\cos(0)\right] = \arcsin(1) = \frac{\pi}{2}$

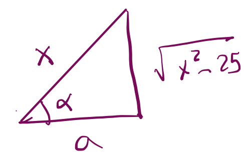
$\tan\left[\arcsin(1)\right] = \tan\left(\frac{\pi}{2}\right)$ Undefined!

$\sin\left[\arctan(-\frac{\sqrt{3}}{3})\right] = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

12. Write the following as an equivalent function of x:

$$\tan\Big[\underbrace{\arcsin\left(\frac{\sqrt{x^2-25}}{x}\right)}_{\alpha}\Big] = \tan\alpha = \frac{\sqrt{x^2-25}}{5}$$

$$\arcsin\left(\frac{\sqrt{x^2-25}}{x}\right) = \alpha \iff \frac{\sqrt{x^2-25}}{x} = \sin\alpha \qquad \frac{\text{opp}}{\text{hyp.}}$$



$$x^2 = a^2 + \left(\sqrt{x^2-25}\right)^2$$

$$x^2 = a^2 + x^2 - 25$$

$$0 = a^2 - 25$$

$$\boxed{a = +5}$$

Since α is in QI or QIV

(bc of range of arcsine)

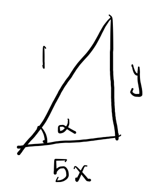
Now $\tan\alpha = \frac{\text{opp.}}{\text{adj.}} = \frac{\sqrt{x^2-25}}{5}$

Domain

$x \in (-\infty, -5] \cup [5, +\infty)$

$$\cot[\underbrace{\arccos(5x)}_{\alpha}] = \cot(\alpha) = \frac{5x}{\sqrt{1-25x^2}} \qquad \frac{\text{adj}}{\text{opp}}$$

$$\alpha = \arccos(5x) \iff \cos(\alpha) = \frac{5x}{1} \qquad \frac{\text{adj}}{\text{hyp.}}$$



$$y^2 + 25x^2 = 1$$

$$y^2 = 1 - 25x^2$$

$$y = +\sqrt{1-25x^2}$$

Since α is in QI or QII

(bc of range of arccosine)

Domain

$x \in \left(-\frac{1}{5}, \frac{1}{5}\right)$

13. Given $y = 1 + \tan(3x + \pi)$, state the period and give an interval including the fundamental cycle of your function. Sketch the graph.

$B = 3$

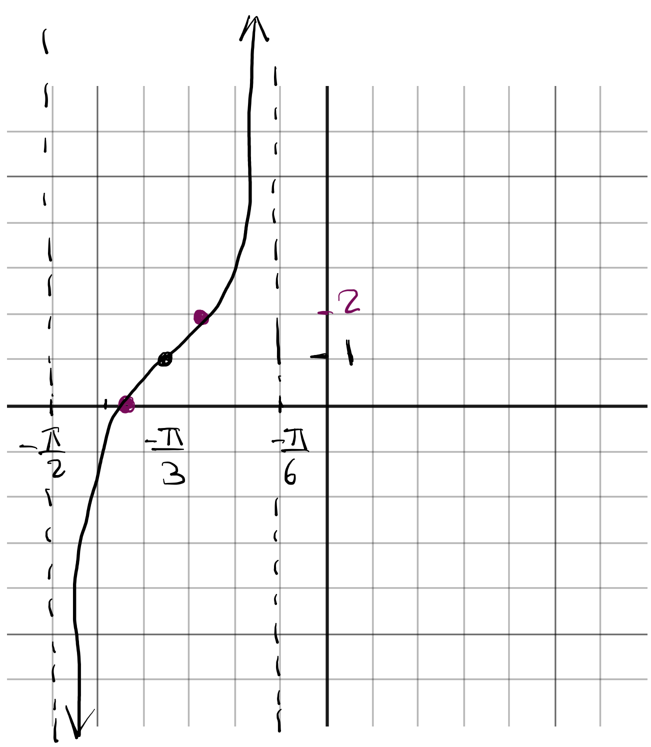
Period $\frac{\pi}{B} = \frac{\pi}{3}$

start of the interval $3x + \pi = -\frac{\pi}{2} \Rightarrow 3x = -\frac{\pi}{2} - \pi$

$3x = \frac{-3\pi}{2} \Rightarrow x = -\frac{\pi}{2}$

End of the interval $\frac{\pi}{3} - \frac{\pi}{2} = \frac{-\pi}{6}$

interval of full cycle $\left[-\frac{\pi}{2}, -\frac{\pi}{6}\right]$



x	$1 + \tan(3x+\pi)$
$-\frac{3\pi}{6} = -\frac{\pi}{2}$	Vert. asy
$-\frac{5\pi}{12}$	$1 - 1 = 0$
$-\frac{2\pi}{6} = -\frac{\pi}{3}$	$1 + 0 = 1$
$-\frac{3\pi}{12}$	$1 + 1 = 2$
$-\frac{\pi}{6}$	vert. asy.

14. Given $y = -7\sin\left(2x - \frac{\pi}{3}\right) + 4$, state the amplitude, period and phase shift of the graph. Sketch the graph.

$A = -7$
$B = 2$
$C = -\frac{\pi}{3}$
$D = 4$

Amplitude : 7

Period : $\frac{2\pi}{2} = \pi$

Phase shift $-\frac{C}{B} = +\frac{\pi}{6}$ (or $2x - \frac{\pi}{3} = 0 \Rightarrow x = \frac{\pi}{6}$)

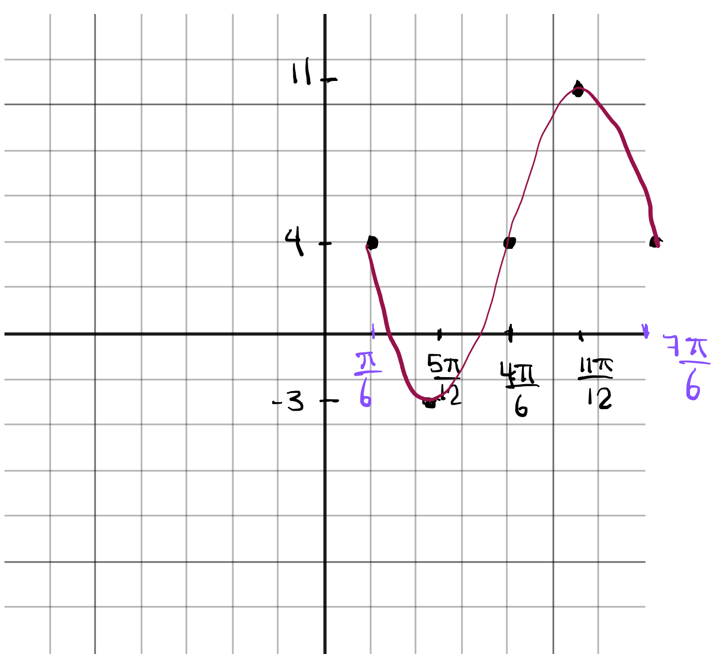
Start : $0 + \frac{\pi}{6}$

End : $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$

Graph one full cycle : $\left[\frac{\pi}{6}, \frac{7\pi}{6}\right]$

key points

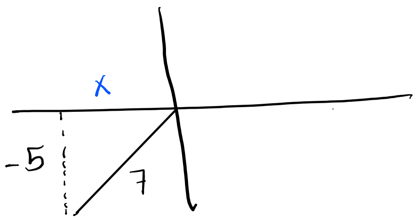
	$-7\sin(2x - \frac{\pi}{3}) + 4$
$\frac{\pi}{6}$	$-7(0) + 4 = 4$
$\frac{5\pi}{12}$	$-7(1) + 4 = -3$
$\frac{4\pi}{6}$	$-7(0) + 4 = 4$
$\frac{11\pi}{12}$	$-7(-1) + 4 = 11$
$\frac{7\pi}{6}$	$-7(0) + 4 = 4$



15. Given $\csc(\theta) = -\dfrac{7}{5}$ and $\tan(\theta) > 0$, find the value of $\sec(\theta)$.

$$\frac{1}{\sin\theta} = -\frac{7}{5} \implies \sin\theta = -\frac{5}{7}$$

$$\left.\begin{array}{l}\tan\theta > 0 \\ \& \ \sin\theta < 0\end{array}\right\} \Rightarrow Q\,\text{III}$$



$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{adj.}} = \frac{-7}{\sqrt{24}}$$

$$x^2 + (-5)^2 = 49$$

$$x^2 = 49 - 25$$

$$x^2 = 24 \qquad x = -\sqrt{24}$$

16. Given t corresponds to the point $\left(\dfrac{1}{5}, -\dfrac{\sqrt{6}}{5}\right)$ on a circle, find the value of $\sin(t)$, $\sec(t)$, and $\tan(t)$.

radius of the circle

$$r^2 = \left(\frac{1}{5}\right)^2 + \left(-\frac{\sqrt{6}}{5}\right)^2 = \frac{1}{25} + \frac{6}{25} = \frac{7}{25}$$

$$r = \frac{\sqrt{7}}{5}$$

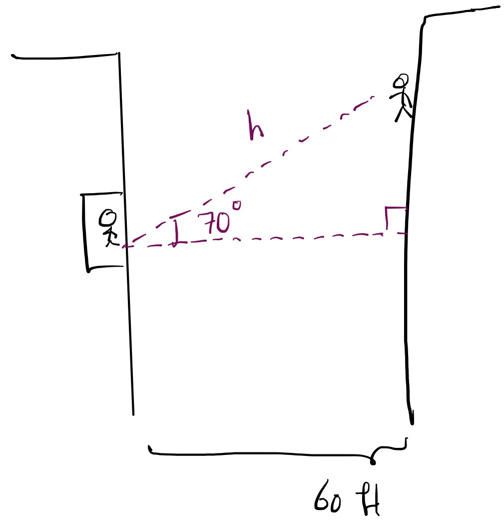
$$\sin(t) = \frac{y}{r} = \frac{-\frac{\sqrt{6}}{5}}{\frac{\sqrt{7}}{5}} = \boxed{-\sqrt{\frac{6}{7}}}$$

$$\sec(t) = \frac{1}{\cos(t)} = \frac{r}{x} = \frac{\sqrt{7}/5}{\frac{1}{5}} = \boxed{\sqrt{7}}$$

$$\tan(t) = \frac{y}{x} = \frac{-\sqrt{6}/5}{\frac{1}{5}} = \boxed{-\sqrt{6}}$$

17. From his hotel room window on the sixth floor, Mike notices some window washers high above him on the hotel across the street. Curious as to their height above the ground, he quickly estimates the buildings are 60 ft apart and the angle of elevation to the workers is 70°. Leave all answers in exact form.

a) How far apart are Mike and the window washers?



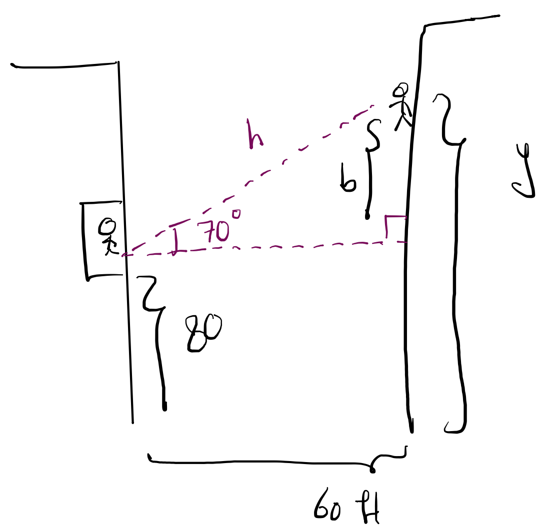
we have adjacent & angle ⇒ want hyp.

$$\cos\theta = \frac{adj}{hyp}$$

$$\cos(70^\circ) = \frac{60}{h}$$

$$h = \frac{60}{\cos(70^\circ)} \quad \text{ft}$$

b) If Mike's hotel floor is 80ft above ground, how far are the window washers from the ground?



$$y = ?$$

$$y = 80 + b$$

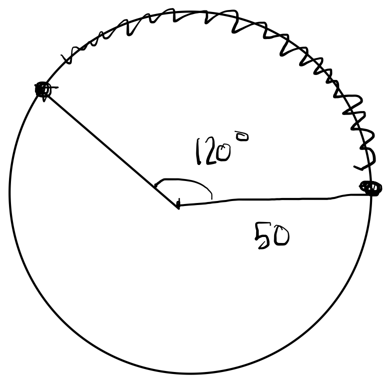
$$\tan(70^\circ) = \frac{opp.}{adj} = \frac{b}{60}$$

$$\Rightarrow b = 60\tan(70^\circ)$$

$$\Rightarrow y = 80 + 60\tan(70^\circ)$$

18. A runner is jogging on a circular track with a radius of 50 meters. The runner starts at one point on the track and runs along the circular path, covering an angle of 120°. What is the distance the runner travels along the track?

$$\theta = 120^\circ \times \frac{\pi}{180^\circ} = \frac{12\pi}{18} = \frac{2\pi}{3}$$



arc length

$$S = r\theta$$

$$S = 50 \times \frac{2\pi}{3} = \frac{100\pi}{3} \text{ m}$$

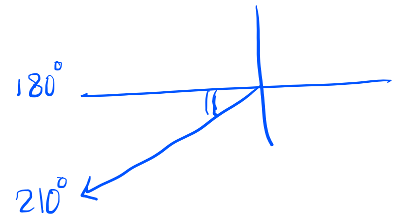
19. Connect each trig or inverse trig function to its correct domain and range.

$\cos(x)$	Domain: $\{t \,\vert\, t \neq \pi k,\ k \text{ is an integer}\}$ Range: $(-\infty, \infty)$
$\sec(x)$	Domain: $(-\infty, -1] \cup [1, \infty)$ Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$\cot(x)$	Domain: $[-1, 1]$ Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\operatorname{arccsc}(x)$	Domain: $(-\infty, \infty)$ Range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\arctan(x)$	Domain: $\left\{t \,\vert\, t \neq \frac{\pi}{2} + \pi k,\ k \text{ is an integer}\right\}$ Range: $(-\infty, -1] \cup [1, \infty)$
$\arcsin(x)$	Domain: $(-\infty, \infty)$ Range: $[-1, 1]$

20. Find the reference angle for:

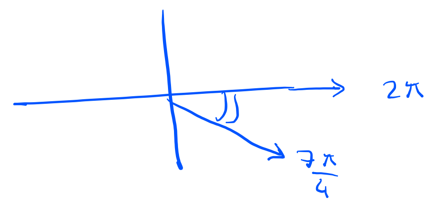
a) $\theta = 210°$

$210° - 180° = \boxed{30°}$



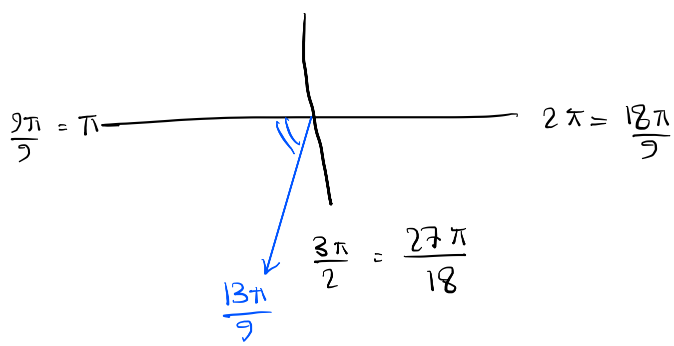
b) $\theta = \dfrac{7\pi}{4}$

$2\pi - \frac{7\pi}{4} = \boxed{\frac{\pi}{4}}$

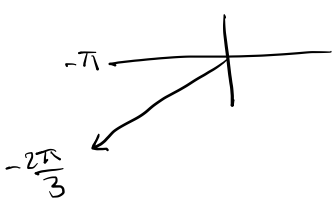


c) $\theta = \dfrac{13\pi}{9} = \dfrac{26\pi}{18}$

$\frac{13\pi}{9} - \pi = \boxed{\frac{4\pi}{9}}$

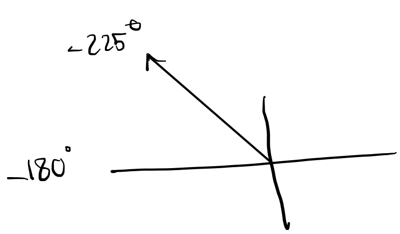


21. Use the reference angle to find the value for $\tan\left(-\frac{2\pi}{3}\right) =$ $+\tan\left(\frac{\pi}{3}\right) = +\sqrt{3}$



$-\frac{2\pi}{3} - (-\pi) = \frac{\pi}{3}$

22. Use the reference angle to find the value for $\cos(-225°) =$ $-\frac{\sqrt{2}}{2}$



$-180° - (-225°) = 45°$

23. Evaluate the following

(a) convert $57°$ to radians.

$$57° \times \frac{\pi}{180°} = \frac{57\pi}{180} = \frac{19\pi}{60}$$

(b) convert $\frac{4\pi}{15}$ radians to degrees.

$$\frac{4\pi}{15} \times \frac{180°}{\pi} = \frac{4 \times 180°}{15} = 4 \times 12 = 48°$$

(c) Supplementary angle for $27°$.

$$\alpha + 27° = 180°$$

$$\alpha = 180° - 27° = 153°$$

(d) Complementary angle for $\frac{2\pi}{7}$

$$\beta + \frac{2\pi}{7} = \frac{\pi}{2}$$

$$\beta = \frac{\pi}{2} - \frac{2\pi}{7} = \frac{3\pi}{14}$$

24. Find all solutions to the system of equations:

(a) $\begin{cases} 3x - 9y = 0 \\ -4x + 12y = 0 \end{cases}$

$\xrightarrow{\times 4}$ $12x - 36y = 0$

$\xrightarrow{\times 3}$ $-12x + 36y = 0$

add +

$0 = 0$ always true $\Rightarrow$ Dependent system

$3x - 9y = 0 \Rightarrow 3x = 9y \Rightarrow x = 3y$

Parametric solution:

free variable y

Parameter t

let $y = t$ $\quad x = 3t$

solutions $(x, y) = (3t, t)$

where t is any real number

(b) $\begin{cases} x^2 + y^2 = 17 \\ y = x + 3 \end{cases}$

Substitution

$x^2 + (x+3)^2 = 17$

$x^2 + x^2 + 6x + 9 = 17$

$2x^2 + 6x - 8 = 0$

$2(\underbrace{x^2 + 3x - 4}) = 0$

$(x+4)(x-1)$

$x_1 = -4 \Rightarrow y_1 = -1$

$x_2 = 1 \Rightarrow y_2 = 4$

Check Solutions:

$\boxed{(x, y) = (-4, -1)}$

$(-4)^2 + (-1)^2 = 16 + 1 = 17$ ✓

$\boxed{(x, y) = (1, 4)}$

$(1)^2 + (4)^2 = 1 + 16 = 17$ ✓

Both will also satisfy $y = x + 3$.

25. Find all solutions for $\cot^2(x) = 0.3\cot(x)\cos(17\pi)$

Note: $\sin(x) \neq 0 \Rightarrow x \neq k\pi$

$\cot^2(x) = 0.3\cot(x) \times (-1)$

$\cot^2 x = -0.3\cot x$

$\cot^2 x + 0.3\cot x = 0$

$\cot x(\cot x + 0.3) = 0$

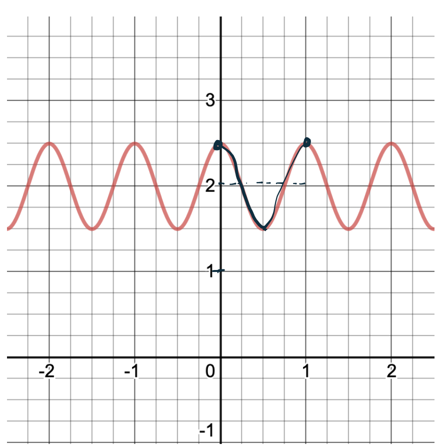
$\cot x = 0$ & $\cot x = -0.3$

$\boxed{x = \frac{\pi}{2} + k\pi}$

$\tan x = -\frac{1}{0.3}$ or $= -\frac{100}{3}$

$\boxed{x = \arctan\left(-\frac{1}{0.3}\right) + k\pi}$

26. Given the graph, write the equation of the cosine function which matches the graph.



Base line: $D = 2$

Amplitude: $2\left(\frac{1}{4}\right) = \frac{1}{2} \Rightarrow A = \frac{1}{2}$

Period: $1 = \frac{2\pi}{B} \Rightarrow B = 2\pi$

Phase shift: $0 = -\frac{C}{B} = \frac{-C}{2\pi} \Rightarrow C = 0$

$y = A\cos(Bx + C) + D$

$y = \frac{1}{2}\cos(2\pi x) + 2$