7.7: MATRIX EXPONENTIALS

Review

- How to **diagonalize** a 2×2 matrix A
  1. Find the eigenvalues λ_1 and λ_2 and eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$.
  2. $A = PDP^{-1}$, where
  $$P = \left[\ \boldsymbol{\xi}^{(1)} \,\middle|\, \boldsymbol{\xi}^{(2)}\ \right], \qquad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$
- **Matrix exponential**
  $$e^{At} = P \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} P^{-1}.$$
- The matrix exponential is useful for solving the initial value problem
  $$\mathbf{x}' = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0.$$
  In particular, the solution is
  $$\mathbf{x}(t) = e^{At}\mathbf{x}_0.$$

Exercise 1

Solve the initial value problem by using the matrix exponential.

$$\mathbf{x}' = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

Find eigenvalues:

$$r^2 + 2r - 3 = 0$$

$$(r+3)(r-1) = 0$$

$$r = -3, 1$$

eigenvector for $r = -3$:

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\vec{\xi} = \vec{0} \Rightarrow \xi_1 + 2\xi_2 = 0 \Rightarrow \xi_1 = -2\xi_2$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -2\xi_2 \\ \xi_2 \end{bmatrix} = \overset{1}{\cancel{\xi_2}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

eigenvector for $r = 1$:

$$\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}\vec{\xi} = \vec{0} \Rightarrow \xi_1 - 2\xi_2 = 0 \Rightarrow \xi_1 = 2\xi_2$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 2\xi_2 \\ \xi_2 \end{bmatrix} = \overset{1}{\cancel{\xi_2}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{t} \end{bmatrix} \left(\frac{-1}{4}\right) \begin{bmatrix} 1 & -2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{t} \end{bmatrix} \left(\frac{-1}{4}\right) \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2e^{-3t} \\ 0 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 4e^{-3t} \\ -2e^{-3t} \end{bmatrix}}$$

7.8: REPEATED EIGENVALUES

Review

- **How to solve** a homogeneous linear system with constant coefficients, $\mathbf{x}' = A\mathbf{x}$.
  1. Assume your solution has the form $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$.
  2. Plug this in to get an eigenvalue problem.
  3. Solve for the eigenvalues.
  4. Based on the eigenvalues:
     - Real distinct eigenvalues:
       (a) Solve for the eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$.
       (b) General solution is $c_1 e^{r_1 t}\boldsymbol{\xi}^{(1)} + c_2 e^{r_2 t}\boldsymbol{\xi}^{(2)}$.
     - Complex eigenvalues:
       (a) Solve for one eigenvector $\boldsymbol{\xi}$.
       (b) Find the real and imaginary parts of the solution $e^{(a+ib)t}\boldsymbol{\xi}$.
       (c) General solution is c_1(real part) $+ c_2$(imaginary part).
     - Repeated eigenvalues:
       (a) Solve for the eigenvector(s).
       (b) If there are two independent eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$:
         (i) General solution is $c_1 e^{rt}\boldsymbol{\xi}^{(1)} + c_2 e^{rt}\boldsymbol{\xi}^{(2)}$.
       (c) If there is only one independent eigenvector $\boldsymbol{\xi}$:
         (i) Solve for the generalize eigenvector $\boldsymbol{\eta}$.
         (ii) General solution is $c_1 e^{rt}\boldsymbol{\xi} + c_2\left(te^{rt}\boldsymbol{\xi} + e^{rt}\boldsymbol{\eta}\right)$.
- The **generalize eigenvector** η can be found via the equation

$$(A - rI)\boldsymbol{\eta} = \boldsymbol{\xi},$$

where r is the eigenvalue and $\boldsymbol{\xi}$ is the eigenvector.

Exercise 2

Find the general solution and sketch the phase portrait.

$$\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}$$

Eigenvalues:

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = 1$$

Eigenvectors for $r=1$:

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow \quad \xi_1 - 2\xi_2 = 0 \Rightarrow \xi_1 = 2\xi_2$$

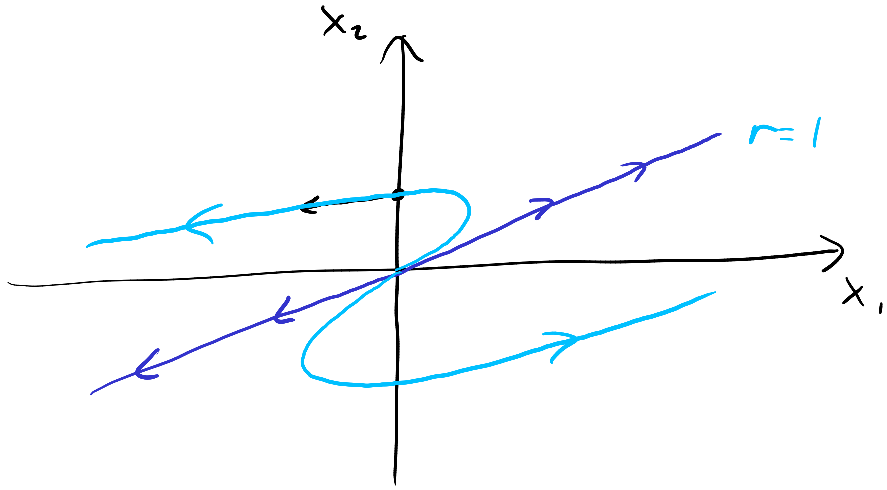
$$\Rightarrow \vec{\xi} = \begin{bmatrix} 2\xi_2 \\ \xi_2 \end{bmatrix} = \overset{1}{\cancel{\xi_2}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Only one independent eigenvector, so we need to find the generalized eigenvector.

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \vec{\eta} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \quad \eta_1 - 2\eta_2 = 1 \Rightarrow \eta_1 = 1 + 2\eta_2$$

$$\vec{\eta} = \begin{bmatrix} 1+2\eta_2 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \overset{0}{\cancel{\eta_2}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

General solution: $\boxed{\vec{x}(t) = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \left(t e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)}$



$$\vec{x} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

The origin is an unstable improper node.

Exercise 3

Find the general solution and sketch the phase portrait.

$$\mathbf{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}$$

Eigenvalues:

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$

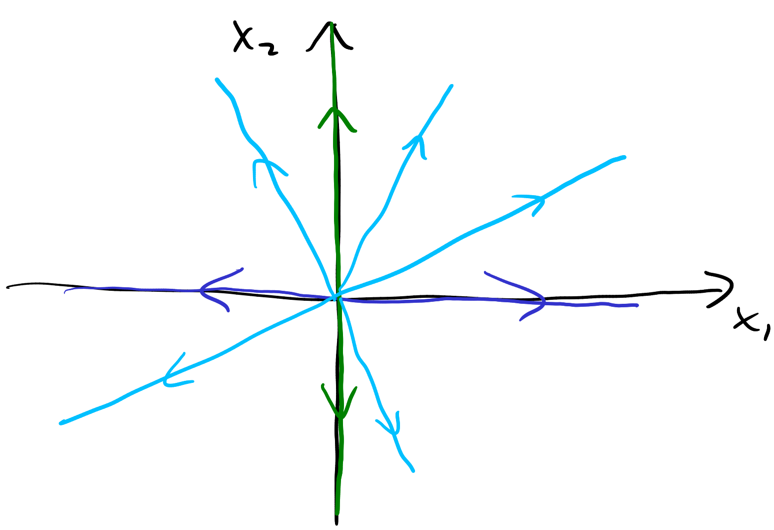
Eigenvectors:

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow 0 = 0$

$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \xi_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \xi_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

two independent eigenvectors

General solution: $\boxed{\vec{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$



The origin is an unstable proper node.

7.9: NONHOMOGENEOUS LINEAR SYSTEMS

Review

- A nonhomogeneous linear system has the form

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t).$$

- There are 4 methods for solving these:

  1. Method of undetermined coefficients
     - Works if $P(t) = A$ and you can guess the particular solution.
  2. Variation of parameters
     - Fundamental matrix: $\Psi(t) = \left[\begin{array}{c|c|c} \mathbf{x}^{(1)} & \cdots & \mathbf{x}^{(n)} \end{array}\right]$.
     - $\mathbf{x}_p(t) = \Psi(t) \int \Psi^{-1}(t)\mathbf{g}(t)\,\mathrm{d}t$.
     - Always works.
  3. Laplace transform
     - Works if $P(t) = A$ and you can take the Laplace transform of everything.
  4. Diagonalization
     - Works if the matrix is diagonalizable.

Exercise 4

Find the general solution using the method of undetermined coefficients.

$$\mathbf{x}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \cos(t) \\ 0 \end{bmatrix}$$

Homogeneous solution:

$$r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm \frac{\sqrt{-4}}{2} = -1 \pm \frac{2i}{2} = -1 \pm i$$

eigenvector for $r = -1 + i$:

$$\begin{bmatrix} 1-(-1+i) & -5 \\ 1 & -3-(-1+i) \end{bmatrix} \vec{\xi} = \vec{0}$$

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow \xi_1 - (2+i)\xi_2 = 0 \Rightarrow \xi_1 = (2+i)\xi_2$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} (2+i)\xi_2 \\ \xi_2 \end{bmatrix} = \overset{1}{\cancel{\xi_2}} \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

$$e^{(-1+i)t} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = e^{-t} e^{it} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = e^{-t}\left(\cos(t) + i\sin(t)\right) \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} 2\cos(t) + 2i\sin(t) + i\cos(t) - \sin(t) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + ie^{-t} \begin{bmatrix} 2\sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}$$

$$\vec{X}_h(t) = c_1 e^{-t}\begin{bmatrix} 2\cos(t)-\sin(t) \\ \cos(t) \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2\sin(t)+\cos(t) \\ \sin(t) \end{bmatrix}$$

Particular solution:

Guess: $\vec{x}_p(t) = \vec{a}\cos(t) + \vec{b}\sin(t)$

$$\vec{x}_p\,'(t) = -\vec{a}\sin(t) + \vec{b}\cos(t)$$

plug into diff eq:

$$-\vec{a}\sin(t) + \vec{b}\cos(t) = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}\left(\vec{a}\cos(t) + \vec{b}\sin(t)\right) + \begin{bmatrix} \cos(t) \\ 0 \end{bmatrix}$$

1. cos terms: $\vec{b} = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}\vec{a} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2. sin terms: $-\vec{a} = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}\vec{b}$

2 $\Rightarrow$ $\vec{a} = -\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}\vec{b}$

plug into 1

$$\vec{b} = -\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}\vec{b} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= -\begin{bmatrix} -4 & 10 \\ -2 & 4 \end{bmatrix}\vec{b} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left(I + \begin{bmatrix} -4 & 10 \\ -2 & 4 \end{bmatrix}\right)\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 10 \\ -2 & 5 \end{bmatrix}\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} -3 & 10 \\ -2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 5 & -10 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2/5 \end{bmatrix}$$

$$\vec{a} = -\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \vec{b}$$

$$= -\begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2/5 \end{bmatrix}$$

$$= -\begin{bmatrix} -1 \\ -1/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1/5 \end{bmatrix}$$

General solution:

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2\sin(t) + \cos(t) \\ \sin(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 1 \\ 1/5 \end{bmatrix} \cos(t) + \begin{bmatrix} 1 \\ 2/5 \end{bmatrix} \sin(t)$$

Exercise 5

Consider the system of differential equations

$$\mathbf{x}' = \frac{1}{t}\begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3t \end{bmatrix}.$$

The general solution to the homogeneous system is

$$\mathbf{x}_h(t) = c_1 t^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 t^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Find a particular solution to the nonhomogeneous system using variation of parameters.

$$\vec{x}^{(1)} = \begin{bmatrix} t^{-1} \\ 2t^{-1} \end{bmatrix} \qquad \vec{x}^{(2)} = \begin{bmatrix} 2t^2 \\ t^2 \end{bmatrix}$$

$$\Psi(t) = [\vec{x}^{(1)} \mid \vec{x}^{(2)}] = \begin{bmatrix} t^{-1} & 2t^2 \\ 2t^{-1} & t^2 \end{bmatrix}$$

$$\Psi(t)^{-1} = \frac{1}{t-4t}\begin{bmatrix} t^2 & -2t^2 \\ -2t^{-1} & t^{-1} \end{bmatrix}$$

$$\vec{x}_p(t) = \Psi(t)\int \Psi^{-1}(t)\,\vec{g}(t)\,dt$$

$$= \Psi(t)\int \frac{-1}{3t}\begin{bmatrix} t^2 & -2t^2 \\ -2t^{-1} & t^{-1} \end{bmatrix}\begin{bmatrix} -2 \\ 3t \end{bmatrix} dt$$

$$= \Psi(t)\int \frac{-1}{3t}\begin{bmatrix} -2t^2 - 6t^3 \\ 4t^{-1} + 3 \end{bmatrix} dt$$

$$= \Psi(t) \int \begin{bmatrix} \frac{1}{3}t + 2t^2 \\ -\frac{4}{3}t^{-2} - t^{-1} \end{bmatrix} dt$$

$$= \begin{bmatrix} t^{-1} & 2t^2 \\ 2t^{-1} & t^2 \end{bmatrix} \begin{bmatrix} \frac{1}{6}t^2 + \frac{2}{3}t^3 \\ \frac{4}{3}t^{-1} - \ln(t) \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} \frac{1}{6}t + \frac{2}{3}t^2 + \frac{8}{3}t - 2t^2\ln(t) \\ \frac{1}{3}t + \frac{4}{3}t^2 + \frac{4}{3}t - t^2\ln(t) \end{bmatrix}}$$

Exercise 6

Solve the initial value problem using the Laplace transform. (Stop when you get to $X(s)$.)

$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \cos(t) \\ t^3 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Laplace transform:

$$s\vec{X}(s) - \vec{x}(0) = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{X}(s) + \begin{bmatrix} \frac{s}{s^2+1} \\ \frac{6}{s^4} \end{bmatrix}$$

Solve for $\vec{X}(s)$:

$$\left(sI - \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\right) \vec{X}(s) = \begin{bmatrix} \frac{s}{s^2+1} \\ \frac{6}{s^4} \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} s-2 & 5 \\ -1 & s+2 \end{bmatrix} \vec{X}(s) = \begin{bmatrix} \frac{s}{s^2+1} + 2 \\ \frac{6}{s^4} + 1 \end{bmatrix}$$

$$\vec{X}(s) = \begin{bmatrix} s-2 & 5 \\ -1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{s}{s^2+1} + 2 \\ \frac{6}{s^4} + 1 \end{bmatrix}$$

$$\boxed{= \frac{1}{(s^2-4)+5} \begin{bmatrix} s+2 & -5 \\ 1 & s-2 \end{bmatrix} \begin{bmatrix} \frac{s}{s^2+1} + 2 \\ \frac{6}{s^4} + 1 \end{bmatrix}}$$

Exercise 7

Find the general solution using diagonalization.

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}}_{A} \mathbf{x} + \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Diagonalize A:

eigenvalues:

$$r^2 - 0r - 1 = 0$$

$$r^2 = 1$$

$$r = \pm 1$$

eigenvector for $r = -1$:

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow \quad 3\xi_1 - \xi_2 = 0 \Rightarrow \xi_2 = 3\xi_1$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ 3\xi_1 \end{bmatrix} = \overset{1}{\cancel{\xi_1}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

eigenvector for $r = 1$:

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \vec{\xi} = \vec{0} \Rightarrow \quad \xi_1 - \xi_2 = 0 \Rightarrow \xi_1 = \xi_2$$

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_2 \\ \xi_2 \end{bmatrix} = \overset{1}{\cancel{\xi_2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$A = PDP^{-1}$, where $P = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\vec{x}' = A\vec{x} + \begin{bmatrix} 3t \\ t \end{bmatrix}$$

$$= PDP^{-1}\vec{x} + \begin{bmatrix} 3t \\ t \end{bmatrix}$$

$$\underbrace{(P^{-1}\vec{x})}_{\vec{y}}' = P^{-1}PD\underbrace{P^{-1}\vec{x}}_{\vec{y}} + P^{-1}\begin{bmatrix} 3t \\ t \end{bmatrix}$$

$$\vec{y}' = D\vec{y} + P^{-1}\begin{bmatrix} 3t \\ t \end{bmatrix}$$

$$\vec{y}' = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\vec{y} + \frac{-1}{2}\begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 3t \\ t \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\vec{y} - \frac{1}{2}\begin{bmatrix} 2t \\ -8t \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\vec{y} + \begin{bmatrix} -t \\ 4t \end{bmatrix}$$

$$\Rightarrow \begin{cases} y_1' = -y_1 - t \\ y_2' = y_2 + 4t \end{cases}$$

Solve for y_1:

$$y_1' + y_1 = -t$$

$$\mu y_1' + \underbrace{\mu}_{\frac{d\mu}{dt}} y_1 = -\mu t$$

$$\frac{d\mu}{dt} = \mu \Rightarrow \mu(t) = e^t$$

$$\frac{d}{dt}\left(e^t y_1(t)\right) = -te^t$$

$$e^t y_1(t) = -\int te^t dt \qquad \begin{matrix} u = t & dv = e^t dt \\ du = dt & v = e^t \end{matrix}$$

$$= -te^t + \int e^t dt$$

$$= -te^t + e^t + c_1$$

$$y_1(t) = -t + 1 + c_1 e^{-t}$$

Solve for $y_2(t)$:

$$y_2' - y_2 = 4t$$

$$\mu y_2' \underbrace{-\mu}_{\frac{d\mu}{dt}} y_2 = 4t\mu$$

$$\frac{d\mu}{dt} = -\mu \Rightarrow \mu(t) = e^{-t}$$

$$\frac{d}{dt}\left(e^{-t} y_2(t)\right) = 4te^{-t}$$

$$e^{-t}y_2(t) = \int 4te^{-t}\,dt \qquad u = 4t \quad dv = e^{-t}dt$$

$$du = 4dt \quad v = -e^{-t}$$

$$= -4te^{-t} + 4\int e^{-t}dt$$

$$= -4te^{-t} - 4e^{-t} + c_2$$

$$y_2(t) = -4t - 4 + c_2e^{t}$$

Plug back into $\vec{x} = P\vec{y}$ to find $\vec{x}$:

$$\vec{x} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -t + 1 + c_1e^{-t} \\ -4t - 4 + c_2e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} -t + 1 + c_1e^{-t} - 4t - 4 + c_2e^{t} \\ -3t + 3 + 3c_1e^{-t} - 4t - 4 + c_2e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} -5t - 3 + c_1e^{-t} + c_2e^{t} \\ -7t - 1 + 3c_1e^{-t} + c_2e^{t} \end{bmatrix}$$

$$\boxed{= t\begin{bmatrix} -5 \\ -7 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} + c_1e^{-t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$