2.6: EXACT DIFFERENTIAL EQUATIONS

Review

- The equation $M(x, y) + N(x, y)y' = 0$ is **exact** if
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$
- How to solve an exact equation:
  1. Define $\Psi_x = M$ and $\Psi_y = N$.
  2. Integrate to find $\Psi(x, y)$.
  3. The equation becomes $\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x, y(x)) = 0$.
  4. Integrate both sides and, if possible, solve for y.

- To find an integrating factor to make an equation exact:
  1. If μ depends only on x,
$$\mu_x = \frac{M_y - N_x}{\underbrace{N}}\mu$$
must not depend on y
  2. If μ depends only on y,
$$\mu_y = \frac{N_x - M_y}{\underbrace{M}}\mu$$
must not depend on x

Exercise 1

Determine if the equation is exact. If it is exact, find the general solution

$$\underbrace{2x+4y}_{M}+\underbrace{(2x-2y)}_{N}y'=0.$$

$$\frac{\partial M}{\partial y}=4\neq\frac{\partial N}{\partial x}=2$$

The equation is not exact.

Exercise 2

Find the value of a for which the equation is exact.

$$\underbrace{3x^2-axy+2}_{M}+\underbrace{(6y^2-x^2+3)}_{N}y'=0.$$

$$M_y=-ax=N_x=-2x$$

$$\Rightarrow \boxed{a=2}$$

Exercise 3

Determine if the equation is exact. If it is exact, find the general solution.

$$(2-\ln(x))y' = \frac{y}{x} + 6x, \quad x > 0.$$

$$\underbrace{-\frac{y}{x} - 6x}_{M} + \underbrace{(2-\ln(x))}_{N}y' = 0$$

$$\frac{\partial M}{\partial y} = -\frac{1}{x} = \frac{\partial N}{\partial x} = \frac{-1}{x} \quad \checkmark \text{ exact}$$

$$\Psi_x = \frac{-y}{x} - 6x \qquad \Psi = -y\ln(x) - 3x^2 + c(y)$$

$$\Rightarrow$$

$$\Psi_y = 2 - \ln(x) \qquad \Psi = 2y - y\ln(x) + c(x)$$

$$\Psi(x,y) = -y\ln(x) - 3x^2 + 2y$$

Diff eq becomes:

$$\frac{d}{dx}\Psi(x, y(x)) = 0$$

Integrate both sides:

$$\Psi(x,y) = c$$

$$-y\ln(x) - 3x^2 + 2y = c \Rightarrow \boxed{y = \frac{3x^2 + c}{2 - \ln(x)}}$$

Exercise 4

Solve the initial value problem and determine where the solution is valid.

$$(2y - x)y' = y - 2x, \quad y(1) = 3.$$

$$\underbrace{2x - y}_{M} + \underbrace{(2y - x)}_{N} y' = 0$$

$$M_y = -1 = N_x = -1 \quad \checkmark \text{ exact}$$

$$\psi_x = 2x - y \qquad \psi = x^2 - xy + c(y)$$

$$\Rightarrow$$

$$\psi_y = 2y - x \qquad \psi = y^2 - xy + c(x)$$

$$\Rightarrow \psi(x,y) = x^2 - xy + y^2$$

General solution:

$$\psi(x,y) = c$$

$$y^2 - xy + x^2 - c = 0$$

$$y = \frac{x \pm \sqrt{x^2 - 4(x^2 - c)}}{2} = \frac{x \pm \sqrt{-3x^2 + 4c}}{2}$$

Use IC to solve for c:

$$y(1) = 3 = \frac{1 \pm \sqrt{-3 \cdot 1^2 + 4c}}{2} = \frac{1 \pm \sqrt{4c - 3}}{2}$$

$$\Rightarrow 6 = 1 \pm \sqrt{4c - 3}$$

$$5 = \pm \sqrt{4c - 3}$$

$$\Rightarrow 25 = 4c - 3$$

$$28 = 4c$$

$$c = 7$$

$$\boxed{y(x) = \frac{x \pm \sqrt{28 - 3x^2}}{2}}$$

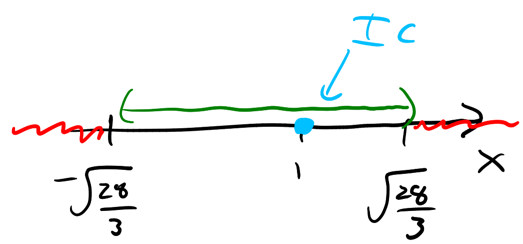
Where is solution valid?

$$28 - 3x^2 \geq 0$$

$$28 \geq 3x^2$$

$$x^2 \leq \frac{28}{3}$$

$$-\sqrt{\frac{28}{3}} \leq x \leq \sqrt{\frac{28}{3}}$$



$$\boxed{\text{Solution is defined on } \left[-\sqrt{\tfrac{28}{3}}, \sqrt{\tfrac{28}{3}}\right].}$$

Exercise 5

Find an integrating factor that makes the following equation exact,

$$\underbrace{1}_{M} + \underbrace{\left(\frac{x}{y} - \sin(y)\right)}_{N} y' = 0.$$

$$\mu + \left(\frac{x}{y} - \sin(y)\right)\mu y' = 0$$

μ depends only on x:

$$\mu_x = \frac{M_y - N_x}{N}\mu$$

$$= \frac{0 - \frac{1}{y}}{\underbrace{\frac{x}{y} - \sin(y)}_{\text{can't depend on } y}}\mu$$

μ depends only on y:

$$\mu_y = \frac{N_x - M_y}{M}\mu$$

$$= \frac{\frac{1}{y} - 0}{\underbrace{1}_{\text{does not depend on } x}}\mu$$

$$\frac{d\mu}{dy} = \frac{\mu}{y}$$

$$\int \frac{d\mu}{\mu} = \int \frac{dy}{y}$$

$$\ln|\mu| = \ln|y| + \cancelto{0}{C}$$

$$\boxed{\mu(y) = y}$$

So, $y + (x - y\sin(y))y' = 0$ is exact.

PRACTICE: SOLVING FIRST ORDER ODES

Review

- You do **NOT** need to guess which method to use to solve a 1st order ODE!

- How to determine which method to use:

1. Is the equation **separable**?

   If yes, then use separation of variables.

2. Is the equation **linear**?

   If yes, then use the method of integrating factors.

2'. Is it a Bernoulli equation[1]?

   If yes, then use $v = y^{1-n}$ to get a linear equation.

3. Is the equation **exact**?

   If yes, then use the method for exact equations.

3'. Is it a homogenous equation[2]?

   If yes, then use $v = \frac{y}{x}$ to get a separable equation.

4. If none of the above,[3]

   then try to find an integrating factor μ to make the equation exact.

[1] A Bernoulli equation has the form $y' + p(t)y = q(t)y^n$. Not all instructors cover this. You can find examples of Bernoulli equations in Section 2.4 of the textbook, #23–25.

[2] This is NOT the same as the homogeneous linear equations that are covered in Chapter 3. The terminology is confusing. "Homogeneous equation" here refers to a 1st order ODE that can be written in the form $y' = f(\frac{y}{x})$. Not all instructors cover this. You can find examples of these in Section 2.2 of the textbook, #25–31.

[3] Not all instructors cover making an equation exact by using an integrating factor.

Exercise 6

Find the general solution to the differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}x} = e^x f - 3xe^{e^x}.$$

$$\frac{df}{dx} - e^x f = -3xe^{e^x}$$

$$\frac{df}{dx}\mu \underbrace{- e^x \mu}_{\frac{d\mu}{dx}} f = -3xe^{e^x}\mu$$

$$\frac{d\mu}{dx} = -e^x \mu$$

$$\int \frac{d\mu}{\mu} = -\int e^x dx$$

$$\ln|\mu| = -e^x + \cancel{C}^{\,0}$$

$$\mu(x) = e^{-e^x}$$

Reverse product rule:

$$\int \frac{d}{dx}\left(e^{-e^x} f(x)\right)dx = \int -3\cancel{e^{-e^x}}\, x\cancel{e^{e^x}}\, dx$$

$$e^{-e^x} f(x) = -\frac{3}{2}x^2 + c$$

$$\boxed{f(x) = -\frac{3}{2}x^2 e^{e^x} + ce^{e^x}}$$

Exercise 7

Find the general solution to the differential equation

$$g' + \frac{x^2+1}{g} = 0.$$

$$\frac{dg}{dx} = -\frac{x^2+1}{g}$$

$$\int g\,dg = -\int (x^2+1)\,dx$$

$$\frac{1}{2}g^2 = -\frac{1}{3}x^3 - x + C$$

$$g^2 = -\frac{2}{3}x^3 - 2x + C$$

$$\boxed{g(x) = \pm\sqrt{-\frac{2}{3}x^3 - 2x + C}}$$

Exercise 8

Find the general solution to the differential equation

$$(\sin(x) + x^2e^y - 1)y' + y\cos(x) = -2xe^y.$$

$$\underbrace{2xe^y + y\cos(x)}_{M} + \underbrace{(\sin(x) + x^2e^y - 1)}_{N} y' = 0$$

$$\frac{\partial M}{\partial y} = 2xe^y + \cos(x) = \frac{\partial N}{\partial x} = \cos(x) + 2xe^y \quad \checkmark \text{exact}$$

$$\psi_x = 2xe^y + y\cos(x) \qquad \psi = x^2e^y + y\sin(x) + C(y)$$

$$\Rightarrow$$

$$\psi_y = \sin(x) + x^2e^y - 1 \qquad \psi = y\sin(x) + x^2e^y - y + C(x)$$

$$\psi(x,y) = x^2e^y + y\sin(x) - y$$

General solution:

$$\boxed{x^2e^y + y\sin(x) - y = C}$$

Not possible to solve for y, so leave it in implicit form.

Exercise 9

Find the solution to the initial value problem

$$y' = \frac{1-2x}{y}, \qquad y(1) = -2.$$

$$\int y\,dy = \int (1-2x)\,dx$$

$$\frac{1}{2}y^2 = x - x^2 + c$$

$$y^2 = 2x - 2x^2 + c$$

$$y(x) = \pm\sqrt{2x-2x^2+c}$$

$$y(1) = \pm\sqrt{2\cdot 1 - 2\cdot 1^2 + c} = -2$$

need the minus

$$2 - 2 + c = 4$$

$$c = 4$$

$$\boxed{y(x) = -\sqrt{2x-2x^2+4}}$$

Exercise 10

Find the general solution to

3.1 SECOND ORDER LINEAR ODES

Review

- A **second order linear ODE** has the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

- A second order linear ODE is **homogeneous** if

$$G(t) = 0.$$

- In the next few sections, we are interested in second order homogeneous linear ODEs with **constant coefficients**.

$$ay'' + by' + cy = 0$$

- Process for **solving** a second order homogeneous linear ODE:
  1. Look for solutions of the form $y(t) = e^{rt}$.
  2. Find the characteristic equation.
  3. Find the roots of the characteristic equation.
  4. The general solution is given by
     - Distinct real roots: $c_1 e^{r_1 t} + c_2 e^{r_2 t}$
     - Complex roots: Section 3.3
     - Repeated real roots: Section 3.4
  5. If you have initial conditions, use them to solve for c_1 and c_2.

Exercise 11

Find the general solution to the differential equation

$$y'' - 4y' + 3y = 0.$$

$y(t) = e^{rt}$

$y'(t) = re^{rt}$

$y''(t) = r^2e^{rt}$

$r^2e^{rt} - 4re^{rt} + 3e^{rt} = 0$

$(r^2 - 4r + 3)\underbrace{e^{rt}}_{\neq 0} = 0$

$r^2 - 4r + 3 = 0$

$(r-3)(r-1) = 0$

$r = 1, 3$

$$\boxed{y(t) = c_1 e^{t} + c_2 e^{3t}}$$

Exercise 12

Find the general solution to the differential equation

$$2y'' - y' - 5y = 0.$$

$2r^2 - r - 5 = 0$

$$r = \frac{+1 \pm \sqrt{1 + 4(2)(5)}}{2(2)} = \frac{1 \pm \sqrt{41}}{4}$$

$$\boxed{y(t) = c_1 e^{\frac{1+\sqrt{41}}{4}t} + c_2 e^{\frac{1-\sqrt{41}}{4}t}}$$

Exercise 13

Solve the initial value problem

$$f'' - 2f' - 8f = 0, \quad f(0) = 3, \quad f'(0) = 1.$$

$$r^2 - 2r - 8 = 0$$

$$(r-4)(r+2) = 0$$

$$r = -2, 4$$

$$f(t) = c_1 e^{-2t} + c_2 e^{4t}$$

Use ICs to solve for c_1 and c_2.

$$f(0) = c_1 + c_2 = 3 \quad \Rightarrow \quad c_1 = 3 - c_2$$

$$f'(t) = -2c_1 e^{-2t} + 4c_2 e^{4t}$$

$$f'(0) = -2c_1 + 4c_2 = 1$$

$$\Rightarrow \quad -2(3 - c_2) + 4c_2 = 1$$

$$-6 + 2c_2 + 4c_2 = 1$$

$$6c_2 = 7$$

$$c_2 = 7/6 \qquad c_1 = 3 - c_1 = \frac{18-7}{6} = \frac{11}{6}$$

$$\boxed{f(t) = \frac{11}{6} e^{-2t} + \frac{7}{6} e^{4t}}$$