Example 1 (15.6). *Compute* $\displaystyle\iiint_E x^2(y+z)\,dV$, *where* $E = \overset{x}{[0,2]} \times \overset{y}{[0,3]} \times \overset{z}{[-1,1]}$.

$$\iiint_E x^2(y+z)\,dV = \int_{-1}^{1}\int_0^3\int_0^2 x^2(y+z)\,dx\,dy\,dz$$

$$= \int_{-1}^{1}\int_0^3 (y+z)\cdot\frac{x^3}{3}\Big|_{x=0}^{2}\,dy\,dz$$

$$= \frac{8}{3}\int_{-1}^{1}\int_0^3 y+z\,dy\,dz$$

$$= \frac{8}{3}\int_{-1}^{1}\left[\frac{y^2}{2}+yz\right]_0^3 dz = \frac{8}{3}\int_{-1}^{1}\left(\frac{9}{2}+3z\right)dz$$

$$= \frac{8}{3}\left[\frac{9}{2}z+\frac{3z^2}{2}\right]_{-1}^{1}$$

$$= \frac{8}{3}\left[\left(\frac{9}{2}+\frac{3}{2}\right)-\left(-\frac{9}{2}+\frac{3}{2}\right)\right] = \frac{8}{3}\cdot 9 = 24$$

Example 2 (15.6). *Compute* $\displaystyle\int_1^2\int_0^{3z}\int_0^{\ln x} xe^{-y}\,dy\,dx\,dz$.

$$= \int_1^2\int_0^{3z}\left[-xe^{-y}\right]_{y=0}^{\ln x}dx\,dz$$

$$= \int_1^2\int_0^{3z}\left[-x\cdot\frac{1}{x}+x\right]dx\,dz = \int_1^2\int_0^{3z}(x-1)\,dx\,dz$$

$$= \int_1^2\left[\frac{x^2}{2}-x\right]_{x=0}^{3z}dz$$

$$= \int_1^2\left[\frac{9}{2}z^2-3z\right]dz = \frac{3}{2}z^3-\frac{3}{2}z^2\Big|_1^2$$

$$= (12-6)-\left(\frac{3}{2}-\frac{3}{2}\right)$$

$$= 6$$

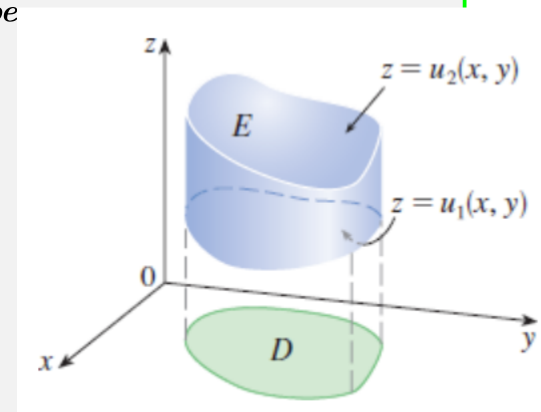
Definition (Type 1 Solid) *A solid region E is of **type 1** if it is of the form*

$$E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$$

where D is the projection of E onto the xy-plane.

In this case,

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$



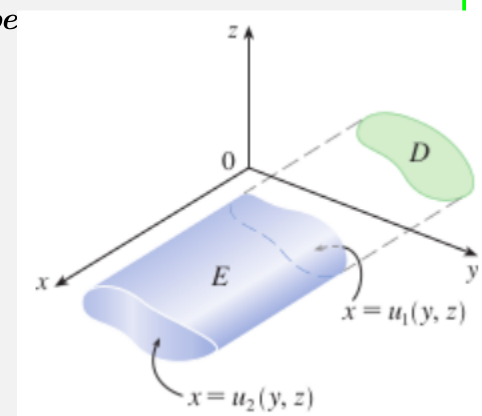
Definition (Type 2 Solid) *A solid region E is of **type 2** if it is of the form*

$$E = \{(x,y,z) \mid (y,z) \in D, u_1(y,z) \le x \le u_2(y,z)\}$$

where D is the projection of E onto the yz-plane.

In this case,

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA$$



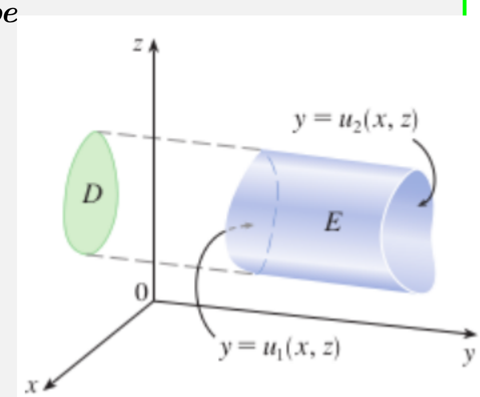
Definition (Type 3 Solid) *A solid region E is of **type 3** if it is of the form*

$$E = \{(x,y,z) \mid (x,z) \in D, u_1(x,z) \le y \le u_2(x,z)\}$$

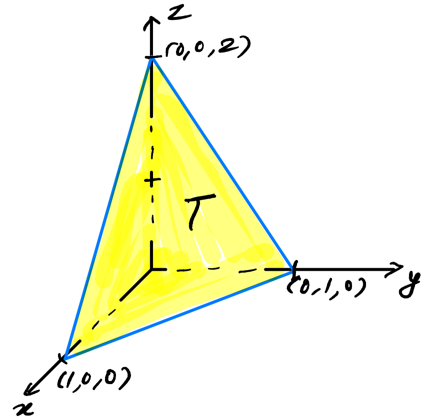
where D is the projection of E onto the xz-plane.

In this case,

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$$



Example 3 (15.6). *Consider the triple integral* $\iiint_T 6y \; dV$*, where* T *is the tetrahedron with vertices at* $(0,0,0), (1,0,0), (0,1,0)$ *and* $(0,0,2)$.



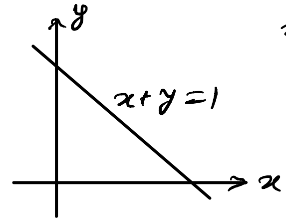
(a) Write the iterated integral with all six possible orders.

The plane passing through $(1,0,0)$, $(0,1,0)$ and $(0,0,2)$ is $2x + 2y + z = 2$.

Projection of T onto xy-plane:

The intersection of $2x+2y+z=2$ and xy-plane is

$2x + 2y = 2 \rightarrow x + y = 1$



Type I

$D = \{(x,y) : 0 \le x \le 1 \quad 0 \le y \le 1-x\}$

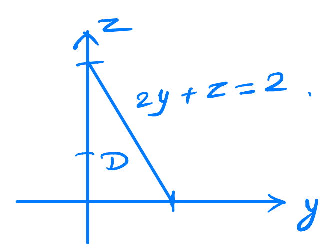
$D = \{(x,y) : 0 \le y \le 1, \; 0 \le x \le 1-y\}$.

Type II

And $2x + 2y + z = 2 \Rightarrow z = 2 - 2x - 2y$. So, $0 \le z \le 2 - 2x - 2y$.

$$\iiint_T 6y \; dV = \int_0^1 \int_0^{1-x} \int_0^{2-2x-2y} 6y \; dz\,dy\,dx$$

$$\iiint_T 6y \; dV = \int_0^1 \int_0^{1-y} \int_0^{2-2x-2y} 6y \; dz\,dx\,dy$$

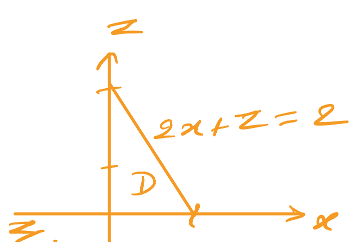


Projection of T onto yz plane $(x=0)$: $2y + z = 2$

$2x + 2y + z = 2 \Rightarrow x = 1 - y - \frac{z}{2} \rightarrow 0 \le x \le 1 - y - \frac{z}{2}$

$$\iiint_T 6y \; dV = \int_0^1 \int_0^{2-2y} \int_0^{1-y-\frac{z}{2}} 6y \; dx\,dz\,dy$$

$$\iiint_T 6y \; dV = \int_0^2 \int_0^{1-\frac{z}{2}} \int_0^{1-y-\frac{z}{2}} 6y \; dx\,dy\,dz$$



Projection of T onto xz plane $(y=0)$: $2x + z = 2$

$2x + 2y + z = 2 \Rightarrow y = 1 - x - \frac{z}{2} \rightarrow 0 \le y \le 1 - x - \frac{z}{2}$.

$$\iiint_T 6y \; dV = \int_0^1 \int_0^{2-2x} \int_0^{1-x-\frac{z}{2}} 6y \; dy\,dz\,dx$$

$$\iiint_T 6y \; dV = \int_0^2 \int_0^{1-\frac{z}{2}} \int_0^{1-x-\frac{z}{2}} 6y \; dy\,dx\,dz$$

(b) Evaluate the integral using one of the six orders.

$$\iiint_T 6y\, dV = 6\int_0^1 \int_0^{1-x} \int_0^{2-2x-2y} y \, dz\, dy\, dx$$

$$= 6\int_0^1 \int_0^{1-x} y\,[2-2x-2y]\, dy\, dx$$

$$= 6\int_0 \int_0 (2-2x)y - 2y^2 \, dy\, dx$$

$$= 6\int_0 \left[(1-x)y^2 - \frac{2}{3}y^3 \right]_0^{1-x} dx$$

$$= 6\int_0 (1-x)^3 \cdot \frac{1}{3}\, dx$$

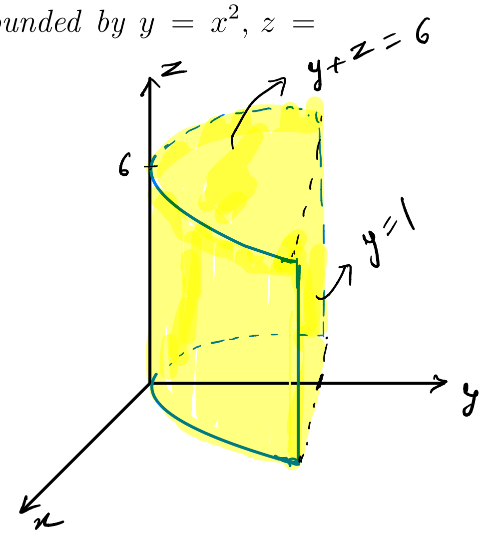
$u = 1-x \Rightarrow du = -dx$

$$= 2\int_0^1 (1-x)^3\, dx$$

$$= -2\int u^3 du = -2 \cdot \frac{u^4}{4}\Big| = -\frac{1}{2}(1-x)^4\Big|_0^1$$

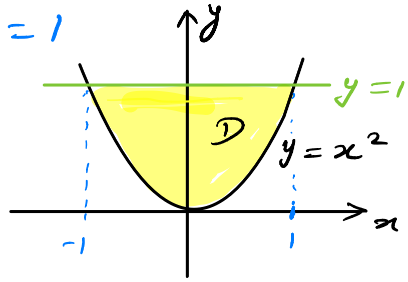
$$= \frac{1}{2}$$

Example 4 (15.6)**.** *Evaluate* $\iiint_E x^2\,dV$, *where* E *is the solid bounded by* $y = x^2$, $z = 0$, $y + z = 6$ *and* $y = 1$.



$y + z = 6 \Rightarrow z = 6 - y$. So, $0 \le z \le 6-y$.

The projection of E onto xy plane is the region bounded by $y = x^2$ and $y = 1$



So,

$E = \{(x,y,z) : -1 \le x \le 1,\ x^2 \le y \le 1,\ 0 \le z \le 6-y\}$.

$$\iiint_E x^2\,dV = \int_{-1}^{1}\int_{x^2}^{1}\int_0^{6-y} x^2\,dz\,dy\,dx$$

$$= \int_{-1}^{1} x^2 \int_{x^2}^{1} (6-y)\,dy\,dx$$

$$= \int_{-1}^{1} x^2 \left[6y - \frac{y^2}{2}\right]_{x^2}^{1} dx$$

$$= \int_{-1}^{1} x^2 \left[\frac{11}{2} - 6x^2 + \frac{x^4}{2}\right] dx$$

$$= \int_{-1}^{1} \frac{11}{2}x^2 - 6x^4 + \frac{x^6}{2}\,dx = \left[\frac{11}{6}x^3 - \frac{6}{5}x^5 + \frac{x^7}{14}\right]_{-1}^{1}$$

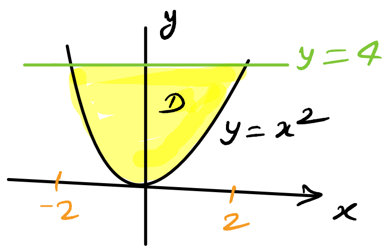
$$= \left(\frac{11}{6} - \frac{6}{5} + \frac{1}{14}\right) - \left(\frac{-11}{6} + \frac{6}{5} - \frac{1}{14}\right)$$

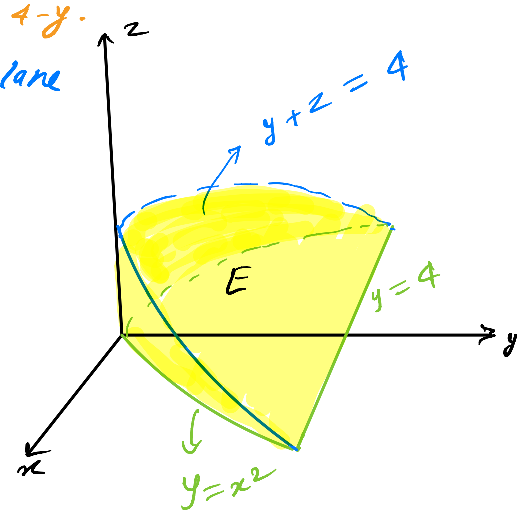
$$= 2\left(\frac{11}{6} - \frac{6}{5} + \frac{1}{14}\right).$$

Example 5 (15.6). *Find the volume of the solid bounded by the parabolic cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 4$.* $\Rightarrow z = 4 - y \rightarrow 0 \le z \le 4 - y$.

The intersection of the plane $y+z=4$ and xy-plane is the line $y=4$. So,

The projection of the solid E onto xy-plane is the region bounded by $y = x^2$ and $y = 4$.





$E = \{(x,y,z) : -2 \le x \le 2,\ x^2 \le y \le 4,\ 0 \le z \le 4-y\}$.

$$V(E) = \iiint_E dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{4-y} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{x^2}^{4} 4-y \, dy\,dx$$

$$= \int_{-2}^{2} 4y - \frac{y^2}{2}\Big|_{y=x^2}^{4} dx$$

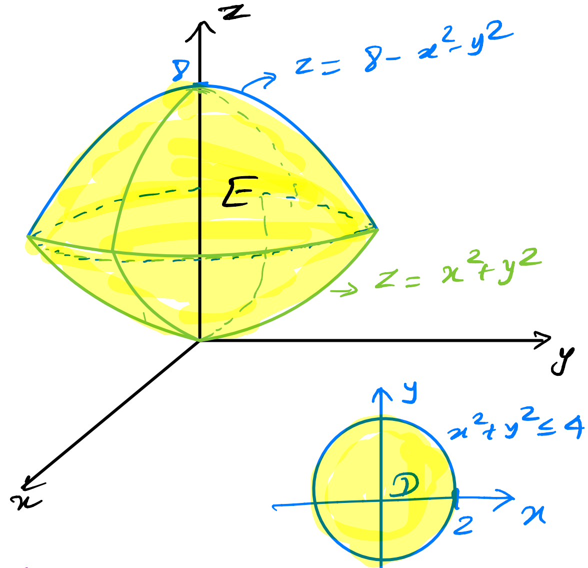
$$= \int_{-2}^{2} 8 - \left(4x^2 - \frac{x^4}{2}\right) dx$$

$$= 8x - \frac{4x^3}{3} + \frac{x^5}{2(5)}\Big|_{-2}^{2}$$

$$= 2\left(16 - \frac{32}{3} + \frac{16}{5}\right)$$

Example 6 (15.6)**.** *Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$*

(a) using a double integral.



The intersection of $z = x^2+y^2$ and $z = 8-x^2-y^2$ is $x^2+y^2 = 8-x^2-y^2$

$$\Rightarrow \quad x^2+y^2 = 4.$$

So, the projection of the solid E onto xy-plane is the circular disk $D: x^2+y^2 \leq 4$.

That is, $D = \{(r,\theta): 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$

Volume $V(E) = \iint_D (\text{top} - \text{bottom})\, dA$

$$= \iint_D [(8-x^2-y^2)-(x^2+y^2)]\, dA = \int_0^{2\pi}\int_0^2 (8-2r^2)\cdot r\, dr\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 8r - 2r^3\, dr\right) = 2\pi\left[4r^2 - \frac{r^4}{2}\right]_0^2 = 16\pi$$

(b) using a triple integral.

$V(E) = \iiint_E dV = \iint_D \left(\int_{x^2+y^2}^{8-x^2-y^2} dz\right) dA$, where D is the projection of E onto xy-plane.

Observe that $D = \{(r,\theta): 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$.

So,

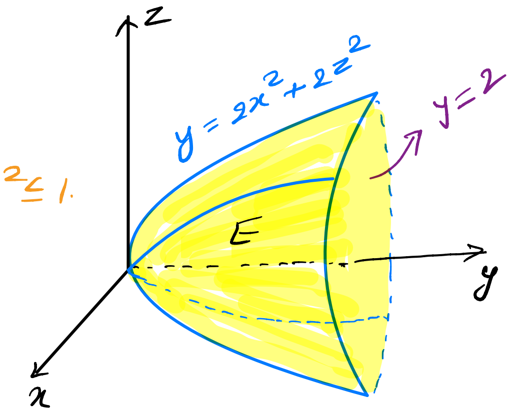
$$V(E) = \iint_D [(8-x^2-y^2) - (x^2+y^2)]\, dA$$

$$= \int_0^{2\pi}\int_0^2 [8 - r^2 - r^2]\cdot r\, dr\, d\theta$$

$$\vdots$$

$$= 16\pi.$$

Example 7 (15.6). *Evaluate* $\iiint_E 2y\,dV$*, where* E *is the solid bounded by the paraboloid* $y = 2x^2 + 2z^2$ *and the plane* $y = 2$.



The intersection of $y = 2x^2 + 2z^2$ and $y = 2$ is

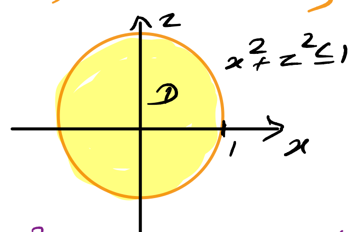
$$x^2 + z^2 = 1$$

So, the projection of E onto xz-plane is $D: x^2 + z^2 \leq 1$.

That is, $D = \{(r, \theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\}$

$$x = r\cos\theta$$

$$z = r\sin\theta$$



So, $E = \{(x, y, z) : 2x^2 + 2z^2 \leq y \leq 2,\ (x, z) \in D\}$.

$$\iiint_E 2y\,dV = \iint_D \left(\int_{2x^2+2z^2}^{2} 2y\,dy \right) dA = \iint_D \left[y^2 \right]_{y=2x^2+2z^2}^{2} dA$$

$$= \iint_D \left[4 - (2x^2 + 2z^2)^2 \right] dA$$

$$= \int_0^{2\pi} \int_0^1 \left[4 - 4r^4 \right] \cdot r\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^1 4r - 4r^5\,dr \right)$$

$$= 2\pi \left[2r^2 - \frac{4r^6}{6} \right]_0^1$$

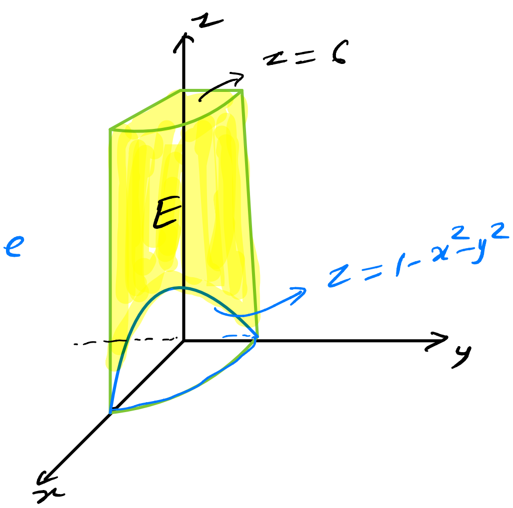
$$= 2\pi \left[2 - \frac{2}{3} \right]$$

$$= \frac{8\pi}{3}$$

Example 8 (15.7). *Evaluate* $\displaystyle\iiint_E 2xz\,dV$, *where E is the solid in the first octant within the cylinder $x^2+y^2=1$, below the plane $z=6$ and above the paraboloid $z=1-x^2-y^2$.*

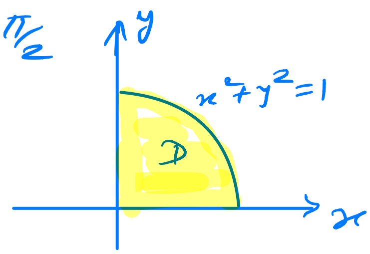
Observe that $1-x^2-y^2 \le z \le 6$.

$\Rightarrow \boxed{1-r^2 \le z \le 6}$



Since the solid E lies within the cylinder $x^2+y^2=1$, the projection of E onto xy-plane is $D: x^2+y^2\le 1,\ x\ge 0,\ y\ge 0$.

That is, $0\le r\le 1,\ 0\le\theta\le \pi/2$



So, $E=\{(r,\theta,z): 0\le r\le 1,\ 0\le\theta\le \frac{\pi}{2},\ 1-r^2\le z\le 6\}$.

$$\iiint_E 2xz\,dV = \int_0^{\pi/2}\int_0^1\int_{1-r^2}^{6} 2(r\cos\theta)z\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^2\cos\theta \cdot z^2\Big|_{z=1-r^2}^{6}\,dr\,d\theta$$

$36-(1+r^4-2r^2) = 35-r^4+2r^2$

$$= \int_0^{\pi/2}\int_0^1 r^2\cos\theta\left[36-(1-r^2)^2\right]dr\,d\theta$$

$$= \left(\int_0^{\pi/2}\cos\theta\,d\theta\right)\left(\int_0^1 35r^2-r^6+2r^4\,dr\right)$$

$$= (1)\cdot\left(\frac{35r^3}{3}-\frac{r^7}{7}+\frac{2r^5}{5}\right)\Big|_0^1$$

$$= \frac{35}{3}-\frac{1}{7}+\frac{2}{5}$$

Example 9 (15.7)**.** *Evaluate the integral by changing into cylindrical coordinates.*

$$I = \int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} \int_{0}^{4-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dx\, dy$$

$E = \{(x, y, z) : -2 \le y \le 2,\ 0 \le x \le \sqrt{4-y^2},\ 0 \le z \le 4 - x^2 - y^2\}$.

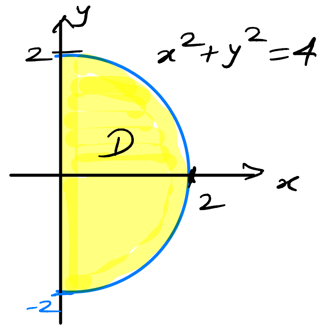
As $z = 4 - x^2 - y^2 = 4 - r^2$, $\boxed{0 \le z \le 4 - r^2.}$

$x = \sqrt{4-y^2} \Rightarrow x^2 + y^2 = 4$

$\Rightarrow r^2 = 4$

$\Rightarrow r = 2$

$\boxed{0 \le r \le 2.}$



And as $x \ge 0$, $\boxed{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$

So, $E = \{(r, \theta, z) : 0 \le r \le 2,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le z \le 4 - r^2\}$.

$$I = \int_{-\pi/2}^{\pi/2} \int_{0}^{2} \int_{0}^{4-r^2} \sqrt{r^2} \cdot r\, dz\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_{0}^{2} r^2 (4 - r^2)\, dr\, d\theta$$

$$= \left(\int_{-\pi/2}^{\pi/2} d\theta \right) \left(\int_{0}^{2} 4r^2 - r^4\, dr \right)$$

$$= \pi \cdot \left[\frac{4}{3} r^3 - \frac{r^5}{5} \right]_0^2 = \pi \left[\frac{32}{3} - \frac{32}{5} \right] = \frac{64\pi}{15}$$

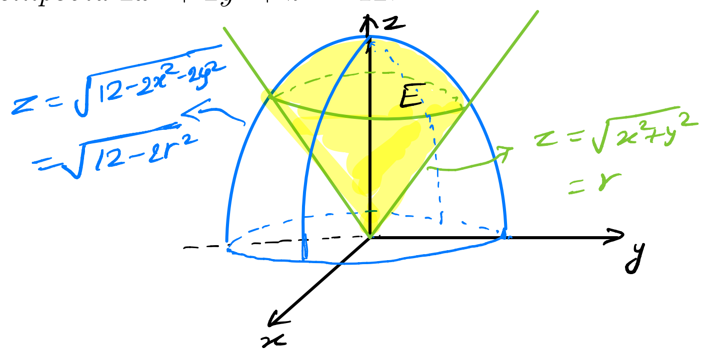
Example 10 (15.7). *(Example 9 from WIR-6) Use cylindrical coordinates to find the volume of the solid between the cone $z = \sqrt{x^2+y^2}$ and the ellipsoid $2x^2+2y^2+z^2=12$.*

$2x^2+2y^2+z^2=12 \Rightarrow z^2 = 12-2x^2-2y^2$

$\Rightarrow z = \sqrt{12-2(x^2+y^2)} = \sqrt{12-2r^2}$

And $z = \sqrt{x^2+y^2} \Rightarrow z = r$.

So, $\boxed{r \le z \le \sqrt{12-2r^2}}$



The intersection of $z=\sqrt{x^2+y^2}$ and $2x^2+2y^2+z^2=12$ is

$$2x^2+2y^2+(x^2+y^2)=12 \Rightarrow x^2+y^2=4.$$

So, the projection of the solid E onto xy-plane is $D: x^2+y^2 \le 4$.

That is, $0 \le r \le 2$, $0 \le \theta \le 2\pi$.

Hence, $E = \{(r,\theta,z) : 0 \le r \le 2, 0 \le \theta \le 2\pi, r \le z \le \sqrt{12-2r^2}\}$.

Volume $V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^2\int_r^{\sqrt{12-2r^2}} r\, dz\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^2 r\left[\sqrt{12-2r^2} - r\right] dr\, d\theta$

$= \int_0^{2\pi}\int_0^2 r\sqrt{12-2r^2}\, dr\, d\theta - \int_0^{2\pi}\int_0^2 r^2\, dr\, d\theta$

$\vdots$

$= \frac{-\pi}{3}\left[8 - (12)^{3/2}\right] - \frac{16\pi}{3}$