Math 151 - Week-In-Review 13

Topics for the week:

5.3 The Fundamental Theorem of Calculus

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1. Use the Fundamental Theorem of Calculus to determine the derivative of $f(x) = \int_0^{\sqrt{x}} (\cos(t))\ dt$.

$$f(x) = \int_0^{\sqrt{x}} (\cos(t))\,dt$$

$$\frac{d f(x)}{dx} = \frac{d}{dx}\left[\int_0^{\sqrt{x}} (\cos(t))\,dt\right]$$

$$= \cos(\sqrt{x}) \cdot \frac{1}{2} x^{-1/2}$$

$$= \frac{\cos(\sqrt{x})}{2\sqrt{x}}$$

2. Compute $\frac{d}{dx}\left[\int_1^{x^4} \left(\sqrt{t^3 + 3t - 1}\right)\ dt\right]$.

$$\frac{d}{dx}\left[\int_1^{x^4} \left(\sqrt{t^3+3t-1}\right) dt\right] = \sqrt{\left((x^4)^3 + 3(x^4) - 1\right)}\, 4x^3$$

$$= 4x^3\sqrt{(x^{12} + 3x^4 - 1)}$$

3. Determine the derivative of $y = \int_{\tan(\theta)}^{4} \left(\sec^5(x) - 6x^3\right)\, dx$.

$$y = \int_{\tan(\theta)}^{4} (\sec^5(x) - 6x^3)\, dx = -\int_{4}^{\tan(\theta)} (\sec^5(x) - 6x^3)\, dx$$

$$\frac{dy}{d\theta} = -\left(\sec^5(\tan(\theta)) - 6(\tan(\theta))^3\right) \cdot \sec^2(\theta)$$

$$= -\sec^2(\theta)\left[\sec^5(\tan(\theta)) - 6\tan^3(\theta)\right]$$

4. Compute the derivative of $\frac{d}{dx}\left[\int_{6x^2}^{5x^4} \left(\frac{2t}{1+t^2}\right) dt\right]$.

$$\frac{d}{dx}\left[\int_{6x^2}^{5x^4} \left(\frac{2t}{1+t^2}\right) dt\right] = \left(\frac{2(5x^4)}{1+(5x^4)^2}\right) \cdot 20x^3 - \left(\frac{2(6x^2)}{1+(6x^2)^2}\right) \cdot 12x$$

$$= 20x^3\left(\frac{10x^4}{1+25x^8}\right) - 12x\left(\frac{12x^2}{1+36x^4}\right)$$

$$= \frac{200x^7}{1+25x^8} - \frac{144x^3}{1+36x^4}$$

5. Compute $\int_0^4 \left(3x - \frac{x^3}{4}\right)\, dx$.

$$\int_0^4 \left(3x - \frac{1}{4}x^3\right) dx = \left(\frac{3x^2}{2} - \frac{1}{16}x^4 + C\right)\Bigg|_0^4$$

$$= \left[\frac{3(4)^2}{2} - \frac{1}{16}(4)^4 + C\right] - \left[\frac{3}{2}(0)^2 - \frac{1}{16}(0)^4 + C\right]$$

$$= \frac{3}{2}(16) - \frac{1}{16}(16)^2 + C - 0 - 0 - C$$

$$= 3(8) - 16$$

$$= 24 - 16$$

$$= 8$$

6. Compute $\int_{\frac{\pi}{2}}^0 \left(\cos^2(\theta)\right)\, d\theta$.

$$\int_{\frac{\pi}{2}}^0 \left(\cos^2(\theta)\right) d\theta = \int_{\frac{\pi}{2}}^0 \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{1}{2}\,\frac{\sin(2\theta)}{2}\right]\Bigg|_{\pi/2}^{0}$$

$$= \left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]\Bigg|_{\pi/2}^{0}$$

$$= \left(\frac{1}{2}(0) + \frac{1}{4}\sin(2(0))\right) - \left(\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin\left(2\left(\frac{\pi}{2}\right)\right)\right)$$

$$= (0 + 0) - \left(\frac{\pi}{4} + 0\right)$$

$$= -\frac{\pi}{4}$$

7. Compute $\int_1^{\sqrt{2}} \left(\frac{x^2+\sqrt{x}}{x^2}\right)\, dx.$

$$\frac{x^{1/2}}{x^2} = x^{1/2-2} = x^{-3/2}$$

$$\int_1^{\sqrt{2}} \left(\frac{x^2+\sqrt{x}}{x^2}\right) dx = \int_1^{\sqrt{2}} \left(1 + x^{-3/2}\right) dx$$

$$= \left[x + \frac{x^{-1/2}}{-1/2}\right]\Big|^{\sqrt{2}}$$

$$= \left[x - \frac{2}{\sqrt{x}}\right]\Big|^{\sqrt{2}}$$

$$= \left[(\sqrt{2}) - \frac{2}{\sqrt{(\sqrt{2})}}\right] - \left[(1) - \frac{2}{\sqrt{(1)}}\right]$$

$$= \sqrt{2} - \frac{2}{\sqrt[4]{2}} - 1 + 2$$

$$= \sqrt{2} - \frac{2}{\sqrt[4]{2}} + 1$$

8. Compute $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (8y^2 - \sin(y))\, dy.$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (8y^2 - \sin(y))\, dy = \left[\frac{8}{3}y^3 + \cos(y)\right]\Big|_{-\pi/2}^{\frac{\pi}{2}}$$

$$= \left[\frac{8}{3}\left(\frac{\pi}{2}\right)^3 + \cos\left(\frac{\pi}{2}\right)\right] - \left[\frac{8}{3}\left(-\frac{\pi}{2}\right)^3 + \cos\left(-\frac{\pi}{2}\right)\right]$$

$$= \left[\frac{8}{3}\left(\frac{\pi^3}{8}\right) + 0\right] - \left[\frac{8}{3}\left(\frac{-\pi^3}{8}\right) + 0\right]$$

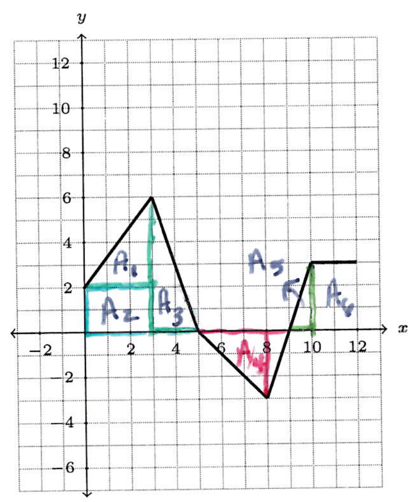
$$= \frac{\pi^3}{3} - \left(-\frac{\pi^3}{3}\right)$$

$$= \frac{2\pi^3}{3}$$

9. Given the initial value problem $\frac{dy}{dx} = \frac{1}{x}$, $y(\pi) = -3$; write a function that will solve this problem.

$\int_1^{\pi} \left(\frac{1}{x}\right) dx$

10. Let $g(x) = \int_0^x (f(t))\, dt$, where $f(x)$ is the function whose graph is shown below.



$A_1 = \frac{1}{2}(3)(4) = 6$ units^2

$A_2 = 3(2) = 6$ units^2

$A_3 = \frac{1}{2}(2)(6) = 6$ units^2

$A_4 = \frac{1}{2}(4)(3) = 6$ units^2

$A_5 = \frac{1}{2}(1)(3) = \frac{3}{2}$ units^2

$A_6 = 2(3) = 6$ units^2

(a) Evaluate $g(5)$

$g(5) = \int_0^5 f(t)dt = A_1 + A_2 + A_3 = 6 + 6 + 6 = 18$ units^2

(b) Evaluate $g(12)$

$g(12) = \int_0^{12} f(t)dt = A_1 + A_2 + A_3 - A_4 + A_5 + A_6 = 6+6+6-6+\frac{3}{2}+6 = 19\frac{1}{2}$ units^2

(c) On what interval is g decreasing?

when $f(x)$ is negative, so on $(5, 9)$

(d) Does $g(x)$ have a maximum, and if so, at what value of x?

yes when $x = 5$

$g(x)$	$\nearrow$	$\searrow$	$\uparrow$
$f(x)$	$+$	$-$	$+$
	5	9	

E1 Some Reminders from Exam 1

11. Given $\mathbf{a} = \langle 2, 5\rangle$ and $\mathbf{b} = 8\mathbf{i} - 6\mathbf{j}$,

$\vec{b}\langle 8,-6\rangle$

(a) Compute $-\mathbf{a} + 3\mathbf{b}$.

$$-\vec{a} + 3\vec{b} = \langle -2,-5\rangle + \langle 24,-18\rangle$$

$$= \langle 22,-23\rangle$$

(b) Compute a vector, $\vec{v}$, in the direction of $\mathbf{a}$, but of length 4.

$$\vec{v} = 4\cdot\frac{\vec{a}}{\|\vec{a}\|}$$

$$|\vec{a}| = \sqrt{(2)^2+(5)^2}$$

$$= \sqrt{4+25}$$

$$= \sqrt{29}$$

$$\vec{v} = 4\cdot\frac{\langle 2,5\rangle}{\sqrt{29}}$$

$$\vec{v} = \left\langle \frac{8}{\sqrt{29}}, \frac{20}{\sqrt{29}}\right\rangle$$

(c) Compute $\mathbf{a}\cdot\mathbf{b}$.

$$\vec{a}\cdot\vec{b} = 2(8) + 5(-6)$$

$$= 16 - 30$$

$$= -14$$

(d) Determine the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$.

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|}$$

$$= \frac{-14}{\sqrt{29}}$$

12. Given a force with magnitude 5 N is applied to an object at a 60° angle, moves the object 3 m; compute the work done.

$W = \vec{F} \cdot \vec{D}$ or $|\vec{F}| \cdot |\vec{D}| \cos(\theta)$

$|\vec{F}| = 5$

$\theta = 60°$

$|\vec{D}| = 3$

$W = 5 \cdot 3 \cos(60°)$

$W = 15 \cdot \left(\frac{1}{2}\right)$ Nm

$W = \frac{15}{2}$ Nm

13. Given $f(x) = \begin{cases} x^2 - 5a & \text{if } x < -1 \\ ax^2 & \text{if } -1 \le x \le 2 \\ 3ax + b & \text{if } x > 2 \end{cases}$, determine the values for a and b that make $f(x)$ a continuous function over $(-\infty, \infty)$.

A, B, C

Check at $x = -1$ and $x = 2$

At $x = -1$

① $f(-1) = a(-1)^2 = a$

② $\lim_{x \to -1} f(x)$

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (x^2 - 5a)$

$= (-1)^2 - 5a$

$= 1 - 5a$

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (ax^2)$

$= a$

So for $\lim_{x \to -1} f(x)$ to exist

$1 - 5a = a$

$1 = 6a$

$\boxed{a = 1/6}$ and $\lim_{x \to -1} f(x) = f(-1)$

at $x = 2$

① $f(2) = a(2)^2 = 4a$

② $\lim_{x \to 2} f(x)$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (ax^2)$

$= 4a$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3ax + b)$

$= 3a(2) + b$

$= 6a + b$

so for $\lim_{x \to 2} f(x)$ to exist

$4a = 6a + b$

$-2a = b$ and we know $a = 1/6$

$-2(1/6) = b$

$\boxed{-1/3 = b}$ and $\lim_{x \to 2} f(x) = f(2)$

14. Compute each limit:

(a) $\lim_{x\to 2^+}\left(\dfrac{x+1}{x^2+2x-8}\right)$ $= \lim_{x\to 2^+}\left(\dfrac{\overbrace{x+1}^{\to 3}}{\underbrace{(x+4)}_{\to 6}\underbrace{(x-2)}_{\to 0^+}}\right)$

$= \infty$

(b) $\lim_{x\to 1^-}\left(\dfrac{x^2+3x-4}{|x-1|}\right)$ $= \lim_{x\to 1^-}\left(\dfrac{(x+4)(x-1)}{-(x-1)}\right)$

$|x-1| = \begin{cases} -(x-1) & \text{if } x<1 \\ x-1 & \text{if } x\geq 1 \end{cases}$

$= \lim_{x\to 1^-}\left(-\underbrace{(x+4)}_{\to 5}\right)$

$= -5$

(c) $\lim_{x\to -\infty}\left(\dfrac{5-4x}{\sqrt{9x^2+2x}}\right)$ $= \lim_{x\to -\infty}\left(\dfrac{\overbrace{-4x+5}^{\to\infty}}{\underbrace{\sqrt{9x^2+2x}}_{\to\infty}}\right)\cdot\left(\dfrac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}}\right)$

Note: $\dfrac{1}{\sqrt{x^2}} = \dfrac{1}{-x}$ as $x\to-\infty$

$= \lim_{x\to -\infty}\left(\dfrac{\frac{-4x}{-x}+\frac{5}{-x}}{\sqrt{\frac{9x^2}{x^2}+\frac{2x}{x^2}}}\right)$

$= \lim_{x\to -\infty}\left(\dfrac{\overbrace{4-5/x}^{\to 4}}{\underbrace{\sqrt{9+2/x}}_{\to 3}}\right)$ $= \dfrac{4}{3}$

(d) $\lim_{x\to\infty}\left(\dfrac{-5e^{-x}+2e^x}{3e^x+8e^{-x}}\right)$

$\lim_{x\to\infty}\left(\dfrac{\overbrace{-5e^{-x}+2e^x}^{\to\infty}}{\underbrace{3e^x+8e^{-x}}_{\to\infty}}\right)\cdot\left(\dfrac{\frac{1}{e^x}}{1/e^x}\right) = \lim_{x\to\infty}\left(\dfrac{\frac{-5e^{-x}}{e^x}+\frac{2e^x}{e^x}}{\frac{3e^x}{e^x}+\frac{8e^{-x}}{e^x}}\right)$

$= \lim_{x\to\infty}\left(\dfrac{\overbrace{-5e^{-2x}+2}^{\to 2}}{\underbrace{3+8e^{-2x}}_{\to 3}}\right)$

$= 2/3$