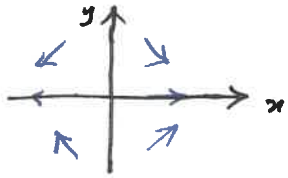
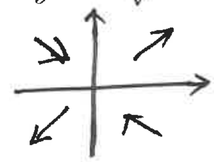
Example 1 (16.1). *Identify the vector field of the vector function* $\mathbf{F}$.

(a) $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$

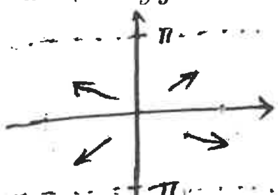


(b) $\mathbf{F}(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}}\,\mathbf{i} + \dfrac{x}{\sqrt{x^2 + y^2}}\,\mathbf{j}$

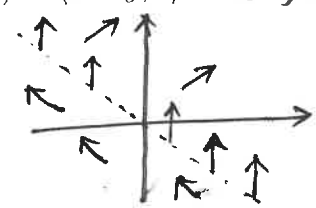
$\|F\| = 1$.

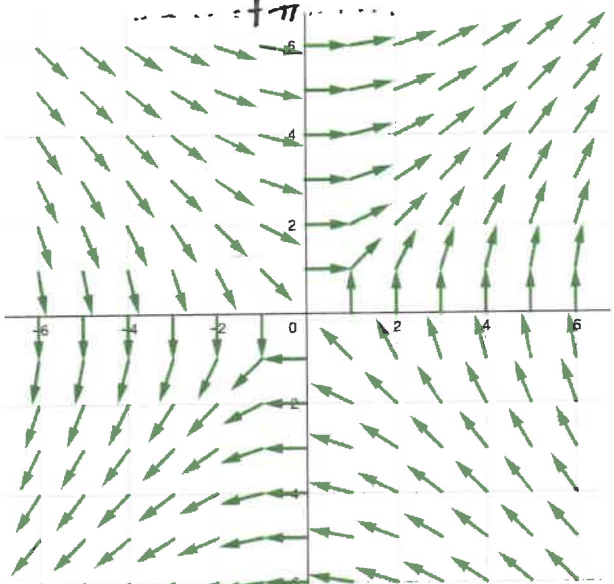


(c) $\mathbf{F}(x, y) = x\,\mathbf{i} + \sin y\,\mathbf{j}$

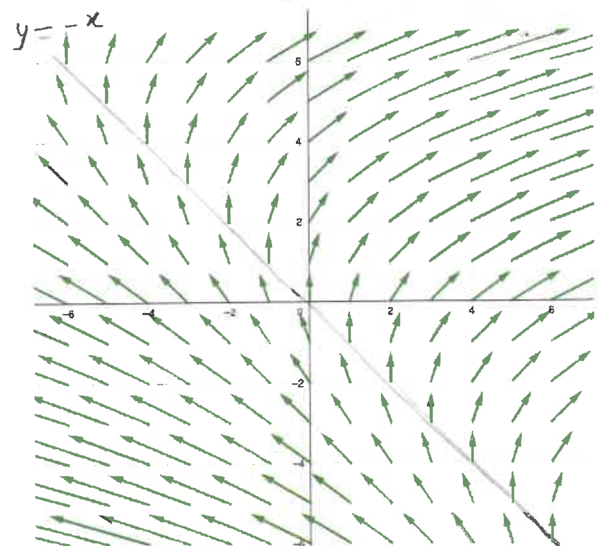


(d) $\mathbf{F}(x, y) = \langle x + y, 3\rangle$ $\quad x + y = 0 \Rightarrow y = -x$

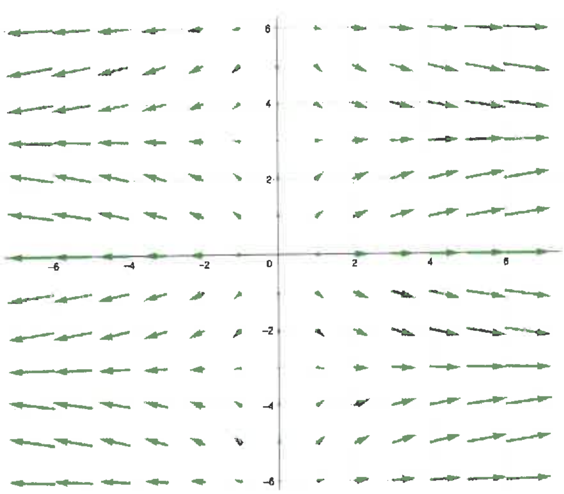




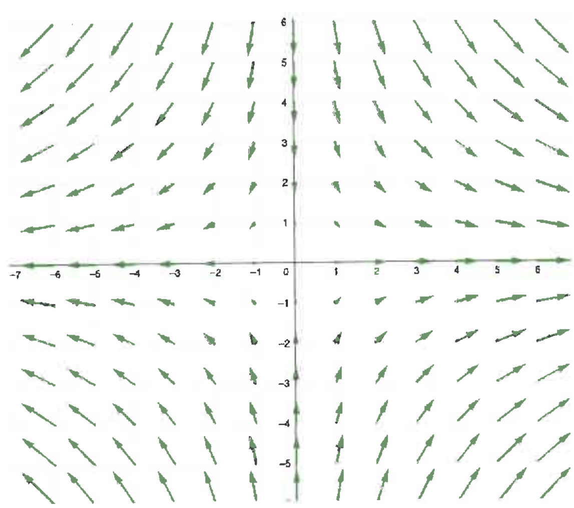
(b)



(d)



(c)



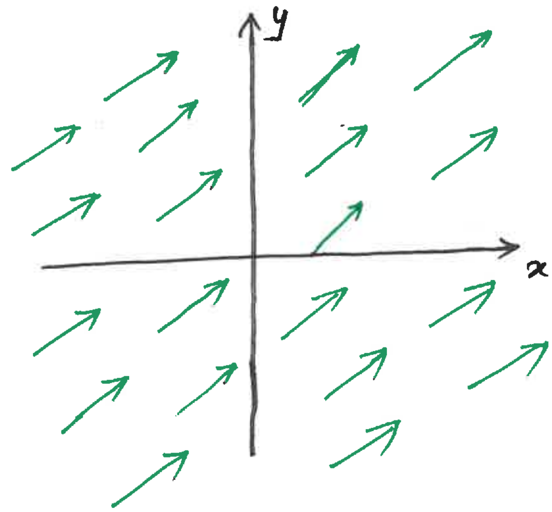
(a)

Example 2 (16.1). *Sketch the gradient vector field of f.*

(a) $f(x, y) = 2x + 3y$

$\nabla f = \langle f_x, f_y \rangle = \langle 2, 3 \rangle$, a constant vector field.

$\|\nabla f\| = \sqrt{13}$.



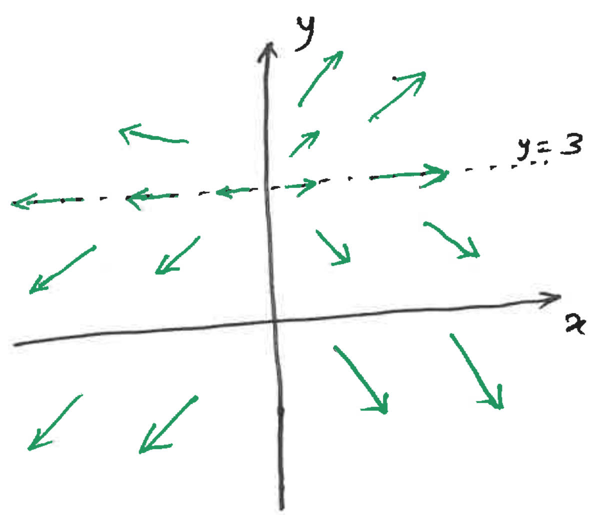
(b) $f(x, y) = \dfrac{x^2}{2} + \dfrac{y^2}{2} - 3y.$

$y - 3 = 0 \Rightarrow y = 3$

$\nabla f(x,y) = \langle x, y - 3 \rangle$

The y-component of the vector is +ve when $y > 3$.

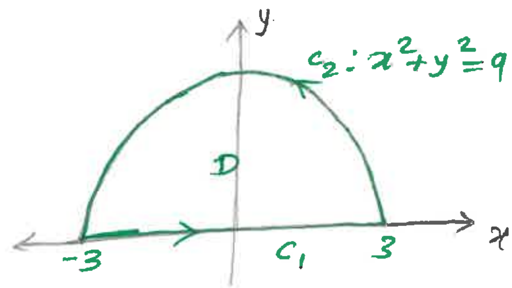
The y-component of the vector is −ve when $y < 3$.



Example 3 (16.2). *Evaluate the line integral*

$$\int_C x^2\,dx + y^2\,dy, \qquad C = C_1 \cup C_2$$

where C is the closed curve oriented counterclockwise in the upper half-plane formed by the circle $x^2 + y^2 = 9$ and line $y = 0$.



Method 1

Over C_1 : $y = 0$, $x = x$, $-3 \le x \le 3$.
$dy = 0$ $\quad dx = dx$

$$\int_{C_1} x^2dx + y^2dy = \int_{-3}^{3} x^2dx + 0$$
$$= \frac{x^3}{3}\Big|_{-3}^{3} = 18$$

Over C_2 : A parametrization of C_2 is $x = 3\cos t$, $y = 3\sin t$, $0 \le t \le \pi$.
$dx = -3\sin t\,dt$, $dy = 3\cos t\,dt$

$$\int_{C_2} x^2dx + y^2dy = \int_0^{\pi} 9\cos^2 t\cdot(-3\sin t)dt + 9\sin^2 t\cdot 3\cos t\,dt$$
$$= -27\int_0^{\pi} \cos^2 t\sin t\,dt + \cancel{27\int_0^{\pi}\sin^2 t\cos t\,dt} \; 0$$
$$= -27\left(-\frac{\cos^3 t}{3}\right)\Big|_0^{\pi} \qquad u = \cos t \Rightarrow du = -\sin t\,dt$$
$$= -18$$

$$\int_C x^2dx + y^2dy = \int_{C_1} x^2dx + y^2dy + \int_{C_2} x^2dx + y^2dy = 18 + (-18) = 0.$$

Method 2 :

$$\int_C \underbrace{x^2}_{P}dx + \underbrace{y^2}_{Q}dy \overset{\text{Green's Theorem}}{=} \iint_D (Q_x - P_y)\,dA = \iint_D 0\,dA = 0.$$

Example 4 (16.2). *Let C be the curve* $\mathbf{r}(t) = \left\langle \overset{x}{2t},\ \overset{y}{t^2},\ \overset{z}{\frac{t^3}{3}} \right\rangle$; $0 \le t \le 2$.

(a) *Evaluate the line integral* $\displaystyle\int_C (xy + 3z)\,ds$.

$r'(t) = \langle 2, 2t, t^2 \rangle$.

$ds = |r'(t)|\,dt = \sqrt{4 + 4t^2 + t^4}\,dt = \sqrt{(2+t^2)^2}\,dt = (2+t^2)\,dt$

$$\int_C (xy+3z)\,ds = \int_0^2 \left(2t \cdot t^2 + 3 \cdot \frac{t^3}{3}\right) \cdot (2+t^2)\,dt$$

$$= \int_0^2 (2t^3 + t^3)(2+t^2)\,dt = \int_0^2 (6t^3 + 3t^5)\,dt$$

$$= \frac{6t^4}{4} + \frac{3t^6}{6}\Big|_0^2 = 24 + 32 = 56$$

(b) *Evaluate the line integral* $\displaystyle\int_C x^2\,dx + y\,dy + 12z\,dz$.

$x = 2t \Rightarrow dx = 2dt$

$y = t^2 \Rightarrow dy = 2t\,dt$

$z = \frac{t^3}{3} \Rightarrow dz = t^2 dt$

$$\int_C x^2 dx + y\,dy + 12z\,dz = \int_0^2 4t^2 \cdot 2dt + t^2 \cdot 2t\,dt + 4t^3\,dt$$

$$= \int_0^2 (8t^2 + 6t^3)\,dt$$

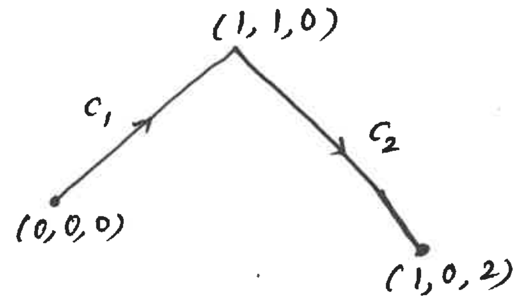
$$= \frac{8}{3}t^3 + \frac{3}{2}t^4\Big|_0^2$$

$$= \frac{64}{3} + 24$$

Example 5 (16.2). *Evaluate the line integral*

$$\int_C (x+y)\,dx + y^2\,dy + z\,dz$$

where C consists of the line segments from $(0,0,0)$ to $(1,1,0)$ and from $(1,1,0)$ to $(1,0,2)$.



Over C_1: $r_0 = (0,0,0)$, $r_1 = (1,1,0)$.

The vector form of the line segment is

$r(t) = r_0 + t(r_1 - r_0)$, $0 \le t \le 1$

$= \langle 0,0,0\rangle + t\langle 1,1,0\rangle$

$= \langle t, t, 0\rangle$.

$x =$, $y = t$, $z = 0$, $0 \le t \le 1$

$dx = dt \quad dy = dt \quad dz = 0$

$$\int_{C_1} (x+y)dx + y^2dy + z\,dz = \int_0^1 2t\,dt + t^2dt + 0 = t^2 + \frac{t^3}{3}\Big|_0^1 = \frac{4}{3}.$$

Over C_2: $r_0 = (1,1,0)$, $r_1 = (1,0,2)$

$r(t) = \langle 1,1,0\rangle + t\langle 1-1, 0-1, 2-0\rangle = \langle 1, 1-t, 2t\rangle$, $0 \le t \le 1$.

$x = 1$, $y = 1-t$, $z = 2t$, $0 \le t \le 1$

$dx = 0 \quad dy = -dt \quad dz = 2\,dt$

$$\int_{C_2} (x+y)\,dx + y^2dy + z\,dz = \int_0^1 (-t)\cdot 0 + (1-t)^2\cdot(-dt) + 2t\cdot 2dt$$

$$= \int_0^1 (-1 - t^2 + 2t + 4t)\,dt = \int_0^1 6t - 1 - t^2\,dt$$

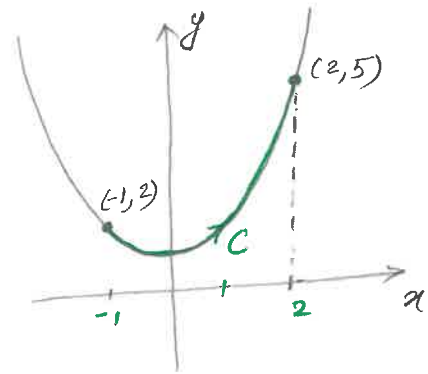
$$= 3t^2 - t - \frac{t^3}{3}\Big|_0^1 = 3 - 1 - \frac{1}{3} = \frac{5}{3}.$$

$$\int_C (x+y)dx + y^2dy + z\,dz = \frac{4}{3} + \frac{5}{3} = 3$$

Example 6 (16.2/16.3). *Evaluate the line integral*

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F}(x,y) = \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{P}\mathbf{i} + \underbrace{\frac{y}{\sqrt{x^2+y^2}}}_{Q}\mathbf{j}$ *and* C *is the part of the parabola* $y = x^2+1$ *from* $(-1,2)$ *to* $(2,5)$.



$C: \ y = x^2+1, \ x = x, \ -1 \le x \le 2.$

$dy = 2x\,dx \quad dx = dx$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \int_C \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy$$

$$= \int_{-1}^{2} \frac{x}{\sqrt{x^2+(x^2+1)^2}}\,dx + \frac{(x^2+1)\cdot 2x\,dx}{\sqrt{x^2+(x^2+1)^2}}$$

$$= \int_{-1}^{2} \frac{2x^3+3x}{\sqrt{x^4+3x^2+1}}\,dx$$

$u = x^4+3x^2+1$

$du = (4x^3+6x)\,dx = 2(2x^3+3x)\,dx$

$$= \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u}\,\Big| = \sqrt{x^4+3x^2+1}\,\Big|_{-1}^{2} = \sqrt{29} - \sqrt{5}$$

Method 2: Note that if $f(x,y) = \sqrt{x^2+y^2}$, then $\nabla f = F$.

So, $\int_C F\cdot dr = \int_C \nabla f\cdot dr \overset{FTLI}{=} f(2,5) - f(-1,2)$

$= \sqrt{4+25} - \sqrt{1+4}$

$= \sqrt{29} - \sqrt{5}$

Example 7 (16.2). *Find the work done by the force field* $\mathbf{F}(x,y,z) = x\,\mathbf{i} - y\,\mathbf{j} + (x+z)\,\mathbf{k}$ *acting along the circular helix* $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\,\mathbf{k}$ *from* $(0,1,\pi)$ *to* $(-1,0,2\pi)$.

$P = x, \quad Q = -y, \quad R = x+z$

$t = \frac{\pi}{2} \Rightarrow r(\frac{\pi}{2}) = (0, 1, \pi)$

$t = \pi \Rightarrow r(\pi) = (-1, 0, 2\pi)$

$x = \cos t \Rightarrow dx = -\sin t\,dt$

$y = \sin t \Rightarrow dy = \cos t\,dt$

$z = 2t \Rightarrow dz = 2\,dt$

$$\text{Work} = \int_C F \cdot dr = \int_C P\,dx + Q\,dy + R\,dz$$

$$= \int_C x\,dx - y\,dy + (x+z)\,dz$$

$$= \int_{\pi/2}^{\pi} \cos t(-\sin t\,dt) - \sin t(\cos t\,dt) + [\cos t + 2t]\,2\,dt$$

$$= \int_{\pi/2}^{\pi} (-\sin 2t + 2\cos t + 4t)\,dt$$

$$= \frac{\cos 2t}{2} + 2\sin t + 2t^2 \Big|_{\pi/2}^{\pi}$$

$$= \left(\frac{1}{2} + 0 + 2\pi^2\right) - \left(\frac{-1}{2} + 2 + \frac{\pi^2}{2}\right)$$

$$= \frac{3\pi^2}{2} - 1$$

☺

Fundamental Theorem of Line Integral (FTLI)

Theorem 2: *Let C be a (piecewise) smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

If C is a smooth curve in $\mathbb{R}^2$ with initial point $A(x_0, y_0)$ and terminal point $B(x_1, y_1)$, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_1, y_1) - f(x_0, y_0)$$

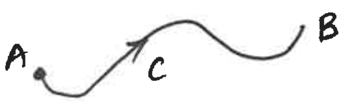
If C is a smooth curve in $\mathbb{R}^3$ with initial point $A(x_0, y_0, z_0)$ and terminal point $B(x_1, y_1, z_1)$, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)$$

If $F = Pi + Qj$ and $\nabla f = F$ for a scalar function f,

then $\displaystyle\int_C F \cdot dr = \int_C \nabla f \cdot dr \overset{FTLI}{=} f(\text{terminal point}) - f(\text{initial point})$

$= f(B) - f(A)$



Theorem 6: *Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D.$$

Then $\mathbf{F}$ is conservative.

Example 8 (16.3). *Determine whether or not* $\mathbf{F}(x,y) = (\sqrt{y}+2xy^2-3)\,\mathbf{i}+\left(\dfrac{x}{2\sqrt{y}}+2x^2y+1\right)\mathbf{j}$ *is conservative. If it is, find a potential function f.*

$$P = \sqrt{y} + 2xy^2 - 3 \Rightarrow P_y = \frac{1}{2\sqrt{y}} + 4xy$$

$$Q = \frac{x}{2\sqrt{y}} + 2x^2y + 1 \Rightarrow Q_x = \frac{1}{2\sqrt{y}} + 4xy.$$

Note that F is defined on the upper half-plane $y>0$, which is simply connected.

As $P_y = Q_x$, F is conservative.

Want $f(x,y)$ such that $\nabla f = F$. That is,

$$f_x = \sqrt{y} + 2xy^2 - 3 \quad (1)$$

$$f_y = \frac{x}{2\sqrt{y}} + 2x^2y + 1 \quad (2)$$

Integrating (1) w.r.t. x,

$$\int f_x(x,y)\,dx = \int (\sqrt{y} + 2xy^2 - 3)\,dx$$

$$f(x,y) = x\sqrt{y} + x^2y^2 - 3x + g(y) \quad (3)$$

$$\cancel{\frac{x}{2\sqrt{y}}} + \cancel{2x^2y} + 1 = f_y = \cancel{\frac{x}{2\sqrt{y}}} + \cancel{2x^2y} + 0 + g'(y)$$

$g'(y) = 1 \Rightarrow g(y) = y + c$. In particular, $g(y) = y$.

So from (3),

$f(x,y) = x\sqrt{y} + x^2y^2 - 3x + y$ is a potential function of F.

Example 9 (16.3). *Find a potential function of the vector field* $\mathbf{F}(x,y,z) = \langle e^{yz},\ xze^{yz},\ xye^{yz}\rangle$, *and use it to find the line integral*

$$\int_C \mathbf{F}\cdot d\mathbf{r},\ \text{where } C:\ x = 2t+1,\ \ y = t^2,\ \ z = \sqrt{t},\quad 0 \le t \le 4.$$

We want to find a function $f(x,y,z)$ such that $\nabla f = F$.

$f_x = e^{yz}$ — 1

$f_y = xze^{yz}$ — 2

$f_z = xye^{yz}$ — 3

Integrating 1 w.r.t. x, $f(x,y,z) = \int e^{yz}\,dx = xe^{yz} + g(y,z)$ — 4

$x\cancel{z}e^{yz} \overset{2}{=} f_y \overset{4}{=} x\cancel{z}e^{yz} + g_y(y,z) \Rightarrow g_y(y,z) = 0$

$\Rightarrow \int g_y(y,z)\,dy = 0$

$\Rightarrow g(y,z) = h(z)$.

So eq^n 4 becomes

$f(x,y,z) = xe^{yz} + h(z)$ — 5

$\cancel{xye^{yz}} \overset{3}{=} f_z = \cancel{xye^{yz}} + h'(z) \Rightarrow h'(z) = 0 \Rightarrow h(z) = c$.

So, $f(x,y,z) = xe^{yz} + c$ is a potential function of F.

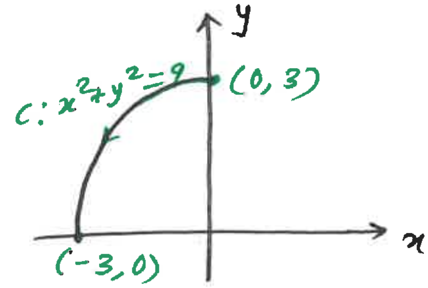
$t = 0 \Rightarrow (x,y,z) = (1,0,0)$

$t = 4 \Rightarrow (x,y,z) = (9,16,2)$

$\int_C F\cdot dr = \int_C \nabla f\cdot dr \overset{FTLI}{=} f(9,16,2) - f(1,0,0)$

$= 9e^{32} - 1$

Example 10 (16.3). *Find the work done by the force field* $\mathbf{F}(x,y) = \langle 2x+y, x+1\rangle$ *acting along the circle* $x^2+y^2=9$ *from* $(0,3)$ *to* $(-3,0)$.



$P = 2x+y \Rightarrow P_y = 1$

$Q = x+1 \Rightarrow Q_x = 1$

$P_y = Q_x$. So, F is conservative.

Want $f(x,y)$ s.t. $\nabla f = F$. That is,

$f_x = 2x+y$ — (1)
$f_y = x+1$ — (2)

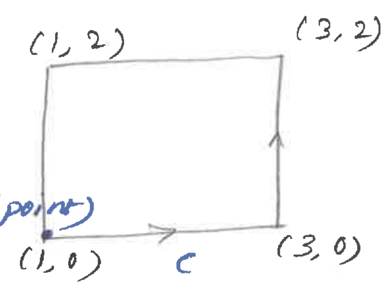
Integrating (1) w.r.t. x, $f(x,y) = \int (2x+y)\,dx = x^2+xy+g(y)$ — (3)

$\cancel{x}+1 \overset{(2)}{=} f_y \overset{(3)}{=} 0+\cancel{x}+g'(y) \Rightarrow g'(y)=1 \Rightarrow g(y)=y+c.$

So, $\boxed{f(x,y) = x^2+xy+y}$ is a potential function of F.

$$\text{Work} = \int_C F\cdot dr = \int_C \nabla f\cdot dr \overset{FTLI}{=} f(-3,0)-f(0,3)$$
$$= 9-3$$
$$= 6$$

Example 11 (16.3). *Find* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F}$ *is a conservative vector field on* $\mathbb{R}^2$ *and* C *is a rectangle with vertices* $(1,0)$, $(3,0)$, $(3,2)$, *and* $(1,2)$.



Let f be such that $\nabla f = F$.

$$\int_C F\cdot dr = \int_C \nabla f\cdot dr \overset{FTLI}{=} f(\text{terminal point}) - f(\text{initial point})$$
$$= 0$$

☺ As C is closed, terminal = initial.