1. Find the local extrema/saddle points for $f(x,y) = x^3 - 3x + 3xy^2$

$\nabla f = \langle f_x, f_y \rangle = \langle 3x^2 - 3 + 3y^2, +6xy \rangle = \langle 0, 0 \rangle$

$f_{xx} = 6x$

$f_{xy} = 6y$

$f_{yy} = 6x$

$\begin{cases} 3x^2 - 3 + 3y^2 = 0 \\ +6xy = 0 \end{cases} \Rightarrow$

$x = 0$: $3y^2 - 3 = 0$, $y^2 = 1$, $y = \pm 1$

or

$y = 0$: $3x^2 - 3 = 0$, $x^2 = 1$, $x = \pm 1$

Critical points: $(1,0), (-1,0), (0,1), (0,-1)$

2nd derivative test.

$$D(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{vmatrix}$$

1. $D(x,y) > 0$ and $f_{xx}(x,y) > 0 \Rightarrow (x,y)$ is a local min
2. $D(x,y) > 0$ and $f_{xx}(x,y) < 0 \Rightarrow (x,y)$ is a local max
3. $D(x,y) < 0$, then $(x,y)$ is a saddle point.

$$D(x,y) = \begin{vmatrix} \boxed{6x} & 6y \\ 6y & 6x \end{vmatrix} = 36x^2 - 36y^2, \quad f_{xx} = 6x$$

$D(1,0) = 36 > 0$, $f_{xx}(1,0) = 6 > 0 \rightarrow$

$D(-1,0) = 36 > 0$, $f_{xx}(-1,0) = -6 < 0 \Rightarrow$

local min @ $(1,0)$

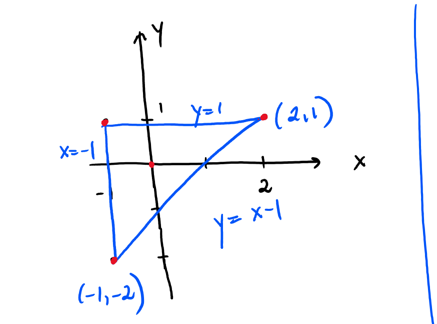
local max @ $(-1,0)$

$D(0,1) = -36 < 0$

$D(0,-1) = -36 < 0$

$\Rightarrow$ $(0, \pm 1)$ saddle points.

2. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + 2xy + 3y^2$ over the set $D$, where $D$ is the closed triangular region with vertices $(-1, 1)$, $(2, 1)$, and $(-1, -2)$.




Critical points inside D:

$$\nabla f = \langle f_x, f_y \rangle = \langle 2x+2y,\ 2x+6y \rangle = \langle 0,0 \rangle$$

$$\begin{cases} \frac{2x+2y}{2} = \frac{0}{2} \\ \frac{2x+6y}{2} = \frac{0}{2} \end{cases} \Rightarrow \quad \begin{aligned} &x+y=0 \Rightarrow x=-y \Rightarrow \boxed{x=0} \\ -\ &x+3y=0 \\ \hline &x-x+(y-3y)=0 \rightarrow -2y=0 \text{ or } \boxed{y=0} \end{aligned}$$

point $\boxed{(0,0)}$

Critical points on the boundary

$\boxed{x=-1}$ $f(x,y) = x^2+2xy+3y^2$, plug in $x=-1 \Rightarrow f(-1,y) = 1-2y+3y^2$

$f'(-1,y) = -2+6y = 0 \Rightarrow y = +1/3$

point $\boxed{(-1, 1/3)}$

$\boxed{y=1}$ $f(x,1) = x^2+2x+3$

$f'(x,1) = 2x+2 = 0 \Rightarrow x=-1$

point $\boxed{(-1,1)}$

$\boxed{y=x-1}$

$$\begin{aligned} f(x,x-1) &= x^2+2x(x-1)+3(x-1)^2 \\ &= x^2+2x^2-2x+3(x^2-2x+1) \\ &= 3x^2-2x+3x^2-6x+3 \\ &= 6x^2-8x+3 \end{aligned}$$

$f'(x,x-1) = 12x-8 = 0 \Rightarrow x = \frac{8}{12} = \frac{2}{3}$

$y = \frac{2}{3} - 1 = -\frac{1}{3}$

point $\boxed{\left(\frac{2}{3}, -\frac{1}{3}\right)}$

$f(x,y) = x^2+2xy+3y^2$

$f(0,0) = \boxed{0}$ abs min value

$f(-1,1) = 1-2+3 = \boxed{2}$

$f(-1, 1/3) = 1 - \frac{2}{3} + 3 \cdot \frac{1}{9} = 1 - \frac{2}{3} + \frac{1}{3} = \boxed{\frac{2}{3}}$

$f\left(\frac{2}{3}, -\frac{1}{3}\right) = \frac{4}{9} + 2 \cdot \frac{2}{3}\left(-\frac{1}{3}\right) + 3 \cdot \frac{1}{9} = \frac{4}{9} - \frac{4}{9} + \frac{1}{3} = \boxed{\frac{1}{3}}$

vertices: $f(2,1) = 4+4+3 = \boxed{11}$

$f(-1,-2) = 1+4+3(4) = \boxed{17}$ abs max value.

3. Find the gradient vector field of the function $f(x, y, z) = xy^2 - yz^3$.

$$\boxed{\nabla f = \langle f_x, f_y, f_z \rangle = \langle y^2, 2xy - z^3, -3yz^2 \rangle}$$

$$\boxed{ds = |\vec{r}\,'(t)|\,dt}$$

$x'(t) = 2\cos t$, $y'(t) = 1$, $z'(t) = -2\sin t$

4. Evaluate the line integral $\int_C x^3 z\,ds$ if $C$ is given by $\boxed{x = 2\sin t}$, $\boxed{y = t}$, $\boxed{z = 2\cos t}$, $0 \leq t \leq \pi/2$.

$\int_C x^3 z\,ds =$

$x = 2\sin t$
$y = t$
$z = 2\cos t$
$ds = |\vec{r}\,'(t)|\,dt$
$= \sqrt{4\cos^2 t + 1 + 4\sin^2 t}\,dt$
$ds = \sqrt{5}\,dt$

$$= \int_0^{\pi/2} (2\sin t)^3 (2\cos t)\sqrt{5}\,dt$$

$$= 16\sqrt{5}\int_0^{\pi/2} \sin^3 t\,\cos t\,dt$$

$u = \sin t$
$du = \cos t\,dt$
$u(0) = \sin 0 = 0$
$u(\pi/2) = \sin\frac{\pi}{2} = 1$

$$= 16\sqrt{5}\int_0^1 u^3\,du$$

$$= \frac{16}{4}\sqrt{5} = \boxed{4\sqrt{5}}$$

5. Evaluate $\int_C y\,dx + z\,dy + x\,dz$ if $C$ consists of the line segments from (0,0,0) to (1,1,2) and from (1,1,2) to (3,1,4).

$C_1$: $(0,0,0) \to (1,1,2)$

parallel to $\langle 1,1,2 \rangle$

$$\begin{cases} x = 0 + 1\cdot t \\ y = 0 + 1\cdot t \\ z = 0 + 2\cdot t \end{cases} \quad \text{or} \quad \begin{bmatrix} x = t \\ y = t \\ z = 2t \end{bmatrix} \begin{bmatrix} dx = dt \\ dy = dt \\ dz = 2\,dt \end{bmatrix}$$

$0 \le t \le 1$

$C_2$: $(1,1,2) \to (3,1,4)$

parallel to $\langle 2,0,2 \rangle$

$$\begin{cases} x = 1 + 2t \\ y = 1 \\ z = 2 + 2t \end{cases} \quad \begin{bmatrix} dx = 2\,dt \\ dy = 0 \\ dz = 2\,dt \end{bmatrix}$$

$\boxed{0 \le t \le 1}$

$$\int_C y\,dx + z\,dy + x\,dz = \int_{C_1} y\,dx + z\,dy + x\,dz + \int_{C_2} y\,dx + z\,dy + x\,dz$$

$$= \int_0^1 \Big[ \underset{y}{t}\,\overset{dx}{(dt)} + \underset{z}{2t}\,\overset{dy}{dt} + \underset{x}{t}\,\overset{dz}{(2dt)} \Big] + \int_0^1 \Big[ \underset{y}{1} \cdot \overset{dx}{2dt} + \overset{z}{(2+2t)} \cdot \overset{dy}{0} + \overset{x}{(1+2t)}\, \overset{dz}{2dt} \Big]$$

$$= \int_0^1 (5t)\,dt + \int_0^1 (4t+4)\,dt = \frac{5t^2}{2}\Big|_0^1 + \Big[\frac{4t^2}{2} + 4t\Big]_0^1$$

$$= \frac{5}{2} + 6 = \boxed{\frac{17}{2}}$$

6. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = x^2y\mathbf{i} + e^y\mathbf{j}$ and $C$ is given by $\mathbf{r}(t) = t^2\mathbf{i} - t^3\mathbf{j}$, $\boxed{0 \le t \le 1.}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$\vec{r}(t) = \langle t^2, -t^3 \rangle$

$\vec{r}\,'(t) = \langle 2t, -3t^2 \rangle$

$\vec{F}(x,y) = \langle x^2y, e^y \rangle$, $x = t^2$; $y = -t^3$

$\vec{F}(\vec{r}(t)) = \langle t^4(-t^3), e^{-t^3} \rangle$

$\vec{F}(\vec{r}(t)) = \langle -t^7, e^{-t^3} \rangle$

$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \langle 2t, -3t^2 \rangle \cdot \langle -t^7, e^{-t^3} \rangle$$
$$= -2t^8 - 3t^2e^{-t^3}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (-2t^8 - 3t^2e^{-t^3})\,dt = -\frac{2t^9}{9}\Big|_0^1 - \int_0^1 3t^2e^{-t^3}\,dt$$

$u = -t^3$

$du = -3t^2\,dt$

$u(0) = -0^3 = 0$

$u(1) = -1^3 = -1$

$$= -\frac{2}{9} + \int_0^{-1} e^u\,du = -\frac{2}{9} - \int_{-1}^{0} e^u\,du = -\frac{2}{9} - e^u\Big|_{-1}^{0}$$

$$= -\frac{2}{9} - e^0 + e^{-1} = -\frac{2}{9} - 1 + \frac{1}{e} = \boxed{\frac{1}{e} - \frac{11}{9}}$$

If $\vec{F}$ is conservative vector field with a potential function $u$, then

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = u(\text{end}) - u(\text{start})}$$

7. Show that $\mathbf{F}(x,y) = (2x + y^2 + 3x^2y)\mathbf{i} + (2xy + x^3 + 3y^2)\mathbf{j}$ is conservative vector field. Use this fact to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ if $C$ is the arc of the curve $y = x\sin x$ from $\underset{\text{start}}{\boxed{(0,0)}}$ to $\underset{\text{end}}{\boxed{(\pi,0)}}$.

conservative $\rightarrow$ $\boxed{\vec{F} = \langle 2x+y^2+3x^2y,\ 2xy+x^3+3y^2\rangle}$

if $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$, and $\boxed{\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}}$, then $\vec{F}$ is conservative

$$\frac{\partial(2x+y^2+3x^2y)}{\partial y} = 2y+3x^2$$

match!

$$\frac{\partial(2xy+x^3+3y^2)}{\partial x} = 2y+3x^2$$

Find a potential function $u(x,y)$ for $\vec{F}$.

Find $u(x,y)$ such that $\nabla u = \vec{F}$

or $\langle u_x, u_y\rangle = \vec{F} = \langle 2x+y^2+3x^2y,\ 2xy+x^3+3y^2\rangle$

$$\begin{cases} \int u_x\,dx = \int(2x+y^2+3x^2y)\,dx \longrightarrow u(x,y) = x^2 + \boxed{xy^2+x^3y} + g(y) \\ \int u_y\,dy = \int(2xy+x^3+3y^2)\,dy \longrightarrow u(x,y) = \boxed{y^2x + x^3y} + y^3 + h(x) \end{cases}$$

$$\boxed{u(x,y) = xy^2 + x^3y + x^2 + y^3}$$

$$\int_C \vec{F}\cdot d\vec{r} = u(\pi,0) - u(0,0)$$
$$= \pi\cdot 0^2 + \pi^3\cdot 0 + \pi^2 + 0^3 - 0 = \boxed{\pi^2}$$

8. Show that $\mathbf{F}(x,y,z) = yz(2x+y)\mathbf{i} + xz(x+2y)\mathbf{j} + xy(x+y)\mathbf{k}$ is conservative vector field. Use this fact to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ if $C$ is given by $\mathbf{r}(t) = (1+t)\mathbf{i} + (1+2t^2)\mathbf{j} + (1+3t^3)\mathbf{k}$, $0 \le t \le 1$.

$\vec{F}$ is conservative if and only if curl $\vec{F} = \vec{0}$

$\vec{F}(x,y,z) = \langle 2xyz + y^2z,\ x^2z + 2xyz,\ x^2y + xy^2 \rangle$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz+y^2z & x^2z+2xyz & x^2y+xy^2 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} \frac{\partial}{\partial y} & \partial/\partial z \\ xz^2+2xyz & x^2y+xy^2 \end{vmatrix} - \vec{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2xyz+y^2z & x^2y+xy^2 \end{vmatrix} + \vec{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xyz+y^2z & xz^2+2xyz \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(x^2y+xy^2) - \frac{\partial}{\partial z}(x^2z+2xyz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(x^2y+xy^2) - \frac{\partial}{\partial z}(2xyz+y^2z)\right] + \vec{k}\left[\frac{\partial}{\partial x}(x^2z+2xyz) - \frac{\partial}{\partial y}(2xyz+y^2z)\right]$$

$$= \vec{i}\left[x^2+2xy - x^2 - 2xy\right] - \vec{j}\left[2xy + y^2 - 2xy - y^2\right] + \vec{k}\left[2xz + 2yz - 2xz - 2yz\right]$$

$$= \langle 0, 0, 0 \rangle$$

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Find a potential function $u(x,y,z)$ for $\vec{F}$: or $\nabla u = \vec{F}$

$$\begin{cases} \int u_x\,dx = \int(2xyz+y^2z)\,dx \longrightarrow u(x,y,z) = \boxed{x^2yz + xy^2z} + f(y,z) \\ \int u_y\,dy = \int(x^2z+2xyz)\,dy \longrightarrow u(x,y,z) = \boxed{x^2yz + xy^2z} + g(x,z) \\ \int u_z\,dz = \int(x^2y+xy^2)\,dz \longrightarrow u(x,y,z) = \boxed{x^2yz + xy^2z} + h(x,y) \end{cases}$$

$$\boxed{u(x,y,z) = x^2yz + xy^2z}$$

Find start and end points.

$\vec{r}(t) = \langle 1+t,\ 1+2t^2,\ 1+3t^3 \rangle$, $0 \le t \le 1$

start $\longrightarrow$ $\vec{r}(0) = \langle 1, 1, 1 \rangle$

end $\longrightarrow$ $\vec{r}(1) = \langle 2, 3, 4 \rangle$

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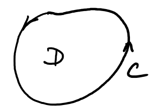
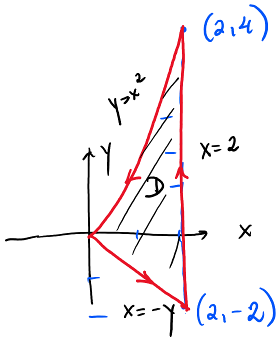
$$\int_C \vec{F}\cdot d\vec{r} = u(2,3,4) - u(1,1,1) = 2^2\cdot 3\cdot 4 + 2\cdot 3^2\cdot 4 - 1 - 1$$

$$= 48 + 72 - 2 = \boxed{118}$$

$$\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$




9. Given the line integral $I = \oint_C 4x^2y\,dx - (2+x)\,dy$ where $C$ consists of the line segment from $(0,0)$ to $(2,-2)$, the line segment from $(2,-2)$ to $(2,4)$, and the part of the parabola $y = x^2$ from $(2,4)$ to $(0,0)$. Use Green's theorem to **evaluate** the given integral and **sketch** the curve $C$ indicating the **positive direction.**



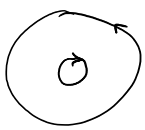

$$\oint_C 4x^2y\,dx - (2+x)\,dy = \iint_D \left[\frac{\partial}{\partial x}(2+x) - \frac{\partial}{\partial y}(4x^2y)\right] dA$$

$$= -\iint_D (1+4x^2)\,dA = -\int_0^2\int_{-x}^{x^2} (1+4x^2)\,dy\,dx$$

$$= -\int_0^2 (1+4x^2)\,y\Big|_{-x}^{x^2} dx = -\int_0^2 (1+4x^2)(x^2+x)\,dx$$

$$= -\int_0^2 (x^2+x+4x^4+4x^3)\,dx = -\left(\frac{x^3}{3}+\frac{x^2}{2}+\frac{4x^5}{5}+\frac{4x^4}{4}\right)_0^2$$

$$= -\left(\frac{8}{3}+2+\frac{128}{5}+16\right) = \ldots$$



$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$, if $\vec{F}(x,y,z) = \langle P, Q, R \rangle$

10. Find curl $\mathbf{F}$ and div $\mathbf{F}$ if $\mathbf{F} = x^2z\mathbf{i} + 2x\sin y\mathbf{j} + 2z\cos y\mathbf{k}$

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(2x\sin y) + \frac{\partial}{\partial z}(2z\cos y) = 2xz + 2x\cos y + 2\cos y$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2z & 2x\sin y & 2z\cos y \end{vmatrix} = \vec{i}\left(\frac{\partial}{\partial y}(2z\cos y) - \cancel{\frac{\partial}{\partial z}(2x\sin y)}^{\,0}\right)$$

$$-\vec{j}\left(\cancel{\frac{\partial}{\partial x}(2z\cos y)}^{\,0} - \frac{\partial}{\partial z}(x^2z)\right)$$

$$+\vec{k}\left(\frac{\partial}{\partial x}(2x\sin y) - \cancel{\frac{\partial}{\partial y}(x^2z)}^{\,0}\right)$$

$$= \vec{i}\,(-2z\sin y) + (x^2)\vec{j} + (2\sin y)\vec{k}$$

$$S.A. = \iint_S 1 \cdot dS = \iint_D |\vec{n}|\, dA \qquad \Big| \quad dS = |\vec{n}|\, dA$$

11. Find the area of the surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \le u \le 1$, $0 \le v \le 2$.

$$\vec{r}(u,v) = \langle u^2, uv, \tfrac{1}{2}v^2 \rangle$$

$$\vec{n} = \vec{r}_u \times \vec{r}_v$$

$$\vec{r}_u = \langle 2u, v, 0 \rangle$$

$$\vec{r}_v = \langle 0, u, v \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = \vec{i}\begin{vmatrix} v & 0 \\ u & v \end{vmatrix} - \vec{j}\begin{vmatrix} 2u & 0 \\ 0 & v \end{vmatrix} + \vec{k}\begin{vmatrix} 2u & v \\ 0 & u \end{vmatrix}$$

$$= v^2\vec{i} - 2uv\vec{j} + 2u^2\vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = \sqrt{(v^2+2u^2)^2} = v^2 + 2u^2$$

$$S.A. = \iint_S 1 \cdot dS = \iint_{\substack{0 \le u \le 1 \\ 0 \le v \le 2}} (v^2+2u^2)\, dA = \int_0^2 \int_0^1 (v^2+2u^2)\, du\, dv$$

$$= \int_0^2 \left(v^2 u + \frac{2u^3}{3}\right)\Big|_0^1 dv = \int_0^2 \left(v^2 + \frac{2}{3}\right) dv = \left(\frac{v^3}{3} + \frac{2}{3}v\right)\Big|_0^2$$

$$= \frac{8}{3} + \frac{4}{3} = \boxed{4}$$

12. Find the area of the part of the paraboloid $\underbrace{z = x^2 + y^2}_{\text{surface}}$ that lies inside the cylinder $\underbrace{x^2 + y^2 = 4}_{\text{parameter}}$. domain.

$z = z(x,y)$, $\vec{n} = \langle z_x, z_y, -1 \rangle$

$\vec{n} = \langle 2x, 2y, -1 \rangle$, $|\vec{n}| = \sqrt{4x^2+4y^2+1}$

$$S.A = \iint_S 1 \cdot dS = \iint_{x^2+y^2 \le 4} |\vec{n}|\, dA = \iint_{x^2+y^2 \le 4} \sqrt{4x^2+4y^2+1}\, dA$$

$x = r\cos\theta$
$y = r\sin\theta$
$dA = r\,dr\,d\theta$
$0 \le r \le 2$
$0 \le \theta \le 2\pi$

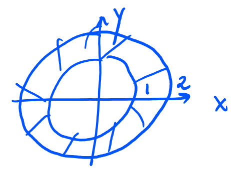
$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2+1}\, r\,dr\,d\theta = \frac{2\pi}{8} \int_0^2 8r\sqrt{4r^2+1}\, dr$$

$u = 4r^2+1$
$du = 8r\,dr$
$u(0) = 4 \cdot 0 + 1 = 1$
$u(2) = 4 \cdot 4 + 1 = 17$

$$= \frac{\pi}{4} \int_1^{17} \sqrt{u}\, du = \frac{\pi}{4} \left. \frac{u^{3/2}}{3/2} \right|_1^{17} = \boxed{\frac{2}{3} \cdot \frac{\pi}{4} \left(17\sqrt{17} - 1\right)}$$

13. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2+y^2}$, $1 \le z \le 4$ if its density function is $\rho(x,y,z) = 10 - z$.

$$m = \iint_S \rho(x,y,z)\,dS = \iint_S (10-z)\,dS$$



$$dS = |\vec{n}|\,dA$$

$$\vec{n} = \langle z_x, z_y, -1\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle$$

$$|\vec{n}| = \sqrt{\frac{x^2+y^2}{x^2+y^2}+1} = \sqrt{2}$$

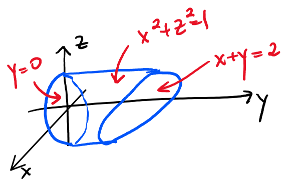
$$m = \iint_D \Big(10 - \underbrace{\sqrt{x^2+y^2}}_{z}\Big)\sqrt{2}\,dA$$

$x = r\cos\theta$
$y = r\sin\theta$
$dA = r\,dr\,d\theta$
$1 \le r \le 2$
$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi}\int_1^2 (10-r)\sqrt{2}\,r\,dr\,d\theta = \sqrt{2}\cdot 2\pi \int_1^2 (10r - r^2)\,dr$$

$$= 2\sqrt{2}\pi\left(\frac{10r^2}{2} - \frac{r^3}{3}\right)\Big|_1^2 = \boxed{2\sqrt{2}\pi\left(20 - 5 - \frac{8}{3} + \frac{1}{3}\right)}$$

14. Evaluate $\oiint_S xy\,dS$ if $S$ is the boundary of the region enclosed by the cylinder $x^2+z^2=1$ and the planes $y=0$ and $x+y=2$.




$$\oiint_S xy\,ds = \iint_{y=0} x\cancel{y}^{\,0}\,ds + \iint_{x+y=2} xy\,ds + \iint_{x^2+z^2=1} xy\,ds$$

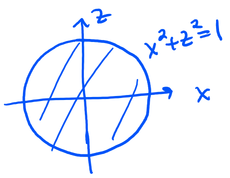
$$\iint_{x+y=2} xy\,ds \quad \left|\begin{array}{l} y=2-x \\ ds=\sqrt{2}\,dA \end{array}\right| = \iint_{x^2+z^2\le 1} x(2-x)\sqrt{2}\,dA \quad \left|\begin{array}{l} x=r\cos\theta \\ z=r\sin\theta \\ 0\le r\le 1 \\ 0\le\theta\le 2\pi \end{array}\right|$$

$y=2-x$

$\vec{n}=\langle y_x, -1, y_z\rangle = \langle -1,-1,0\rangle$

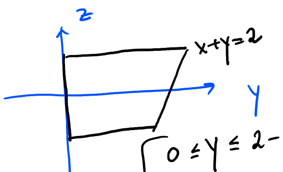
$|\vec{n}|=\sqrt{2}$

parameter domain is $x^2+z^2\le 1$




$$= \int_0^{2\pi}\int_0^1 r\cos\theta\,(2-r\cos\theta)\sqrt{2}\,r\,dr\,d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\int_0^1 \left[2r^2\cos\theta - r^3\cos^2\theta\right] dr\,d\theta$$




$$\left[\begin{array}{l} 0\le y\le 2-x \\ 0\le\theta\le 2\pi \end{array}\right]$$

$$=\sqrt{2}\int_0^{2\pi}\left[\frac{2r^3}{3}\cos\theta - \frac{r^4}{4}\cos^2\theta\right]_0^1 d\theta$$

$$=\sqrt{2}\int_0^{2\pi}\left(\frac{2}{3}\cos\theta - \frac{1}{4}\cos^2\theta\right)d\theta = \sqrt{2}\int_0^{2\pi}\left(\frac{2}{3}\cos\theta - \frac{1}{4}\,\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$=\sqrt{2}\left(\frac{2}{3}\sin\theta - \frac{1}{8}\theta + \frac{1}{16}\sin 2\theta\right)\Big|_0^{2\pi} = -\frac{2\sqrt{2}\,\pi}{8} = \boxed{-\frac{\sqrt{2}\,\pi}{4}}$$

$$\iint_{x^2+z^2=1} xy\,ds \quad \left|\begin{array}{l} x=\cos\theta \\ y=y \\ z=\sin\theta \end{array}\right|$$

$\vec{r}(y,\theta)=\langle \cos\theta, y, \sin\theta\rangle$

$\vec{r}_y=\langle 0,1,0\rangle,\quad \vec{r}_\theta=\langle -\sin\theta, 0, \cos\theta\rangle$

$$\vec{n}=\vec{r}_y\times\vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 0 \\ 0 & \cos\theta \end{vmatrix} - \vec{j}\begin{vmatrix} 0 & 0 \\ -\sin\theta & \cos\theta \end{vmatrix} + \vec{k}\begin{vmatrix} 0 & 1 \\ -\sin\theta & 0 \end{vmatrix}$$

$$=\cos\theta\cdot\vec{i} + \sin\theta\cdot\vec{k}$$

$|\vec{n}|=\sqrt{\cos^2\theta+\sin^2\theta}=1$

$$=\iint_{\substack{0\le y\le 2-x \\ 0\le\theta\le 2\pi}} xy\,dA = \int_0^{2\pi}\int_0^{2-\cos\theta} y\cos\theta\,dy\,d\theta$$

$$=\int_0^{2\pi}\frac{(2-\cos\theta)^2}{2}\cos\theta\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(4-4\cos\theta+\cos^2\theta\right)\cos\theta\,d\theta$$

$\cos^2\theta\cdot\cos\theta = (1-\sin^2\theta)\cos\theta$

$$=\frac{1}{2}\int_0^{2\pi}\left(4\cos\theta - \underbrace{4\cos^2\theta}_{2(1+\cos 2\theta)} + \cos^3\theta\right)d\theta$$

$$\left[2\sin\theta - \left(\theta+\frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} + \frac{1}{2}\cancel{\int_0^{2\pi}(1-\sin^2\theta)\cos\theta\,d\theta}^{\,0}$$

$u=\sin\theta$

$$= \left[2\sin\theta - \left(\theta + \tfrac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} + \cancel{\frac{1}{2}\int_0^{2\pi}(1-\sin^2\theta)\cos\theta\, d\theta}$$

$u = \sin\theta$
$du = \cos\theta\, d\theta$
$u(0) = \sin 0 = 0$
$u(2\pi) = \sin 2\pi = 0$

$$= \boxed{-2\pi}$$

$$\boxed{\iint_S xy\, dS = -\frac{\sqrt{2}\,\pi}{4} - 2\pi}$$

15. Evaluate $\iint_S yz\,dS$ if $S$ is the surface given by $\mathbf{r}(u,v) = \langle uv, u+v, u-v \rangle$, $u^2+v^2 \le 1$.

$dS = |\vec{n}|\,dA$, and $\vec{n} = \vec{r}_u \times \vec{r}_v$

$\vec{r} = \langle uv, u+v, u-v \rangle$

$\vec{r}_u = \langle v, 1, 1 \rangle$

$\vec{r}_v = \langle u, 1, -1 \rangle$

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} v & 1 \\ u & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} v & 1 \\ u & 1 \end{vmatrix}$$

$$= -2\vec{i} - (-v-u)\vec{j} + (-u+v)\vec{k}$$

$$= -2i - (u+v)\vec{j} + (v-u)\vec{k}$$

$$\boxed{\begin{aligned} |\vec{n}| &= \sqrt{4+(u+v)^2+(v-u)^2} \\ &= \sqrt{4+u^2+2uv+v^2+v^2-2uv+u^2} \\ &= \sqrt{4+2u^2+2v^2} \end{aligned}}$$

$$\iint_S yz\,dS = \iint_{u^2+v^2\le 1} (u+v)(u-v)\sqrt{4+2u^2+2v^2}\,dA$$

$$= \iint_{u^2+v^2\le 1} (u^2-v^2)\sqrt{4+2u^2+2v^2}\,dA$$

$u = r\cos\theta$

$v = r\sin\theta$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$u^2+v^2 \le 1$

$0 \le \theta \le 2\pi$

$dA = r\,dr\,d\theta$

$$= \int_0^{2\pi} \int_0^1 \left(r^2\cos^2\theta - r^2\sin^2\theta\right)\sqrt{4+2r^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^3\left(\cos^2\theta - \sin^2\theta\right)\sqrt{4+2r^2}\, dr\,d\theta$$

$$= \int_0^{2\pi} (\underbrace{\cos^2\theta - \sin^2\theta}_{\cos 2\theta})\, d\theta \int_0^1 r^3\sqrt{4+2r^2}\, dr$$

$$= \int_0^{2\pi} \cos 2\theta \, d\theta \int_0^1 r^3\sqrt{4+2r^2}\, dr$$

$$= \underbrace{\tfrac{1}{2}\sin 2\theta \Big|_0^{2\pi}}_{0} \int_0^1 r^3\sqrt{4+2r^2}\, dr = \boxed{0}$$

16. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, if

(a) $\mathbf{F}(x,y,z) = \langle x^2y, -3xy^2, 4y^3 \rangle$ and $S$ is the part of the elliptic paraboloid $z = x^2 + y^2 - 9$ that lies below the rectangle $0 \le x \le 2$, $0 \le y \le 1$ and has downward orientation.

(b) $\mathbf{F}(x,y,z) = \langle x, y, 5 \rangle$ and $S$ is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$.

(a) $\iint_S \vec{F} \cdot d\vec{S}$, $\vec{F} = \langle x^2y, -3xy^2, 4y^3 \rangle$

$S: \ z = x^2 + y^2 - 9$

parameter domain $\quad D: \begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le 1 \end{array}$

$\vec{n} = \langle z_x, z_y, -1 \rangle$ downward orientation

$= \langle 2x, 2y, -1 \rangle$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\substack{0 \le x \le 2 \\ 0 \le y \le 1}} \vec{F} \cdot \vec{n} \, dA$$

$$\vec{F} \cdot \vec{n} = \langle x^2y, -3xy^2, 4y^3 \rangle \cdot \langle 2x, 2y, -1 \rangle$$

$$= 2x^3y - 6xy^3 - 4y^3$$

$$= \int_0^2 \int_0^1 (2x^3y - 6xy^3 - 4y^3) \, dy \, dx$$

$$= \int_0^2 \left( x^3y^2 - \frac{3}{2}xy^4 - y^4 \right)\Big|_0^1 dx$$

$$= \int_0^2 \left( x^3 - \frac{3}{2}x - 1 \right) dx$$

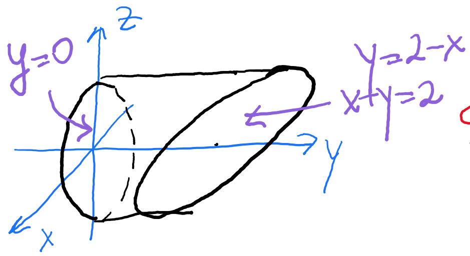
$$= \left(\frac{x^4}{4} - \frac{3x^2}{4} - x\right)\Big|_0^2 = 4 - 3 - 2 = \boxed{-1}$$

---

(b) $\oint\!\!\oint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = \langle x, y, 5\rangle$ and

$S$ is enclosed by $x^2+z^2=1$, $y=0$ and $x+y=2$.

S is closed, so I can use the Divergence Theorem.




$$\oint\!\!\oint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(5) = 2$$

$$\oint\!\!\oint_S \vec{F}\cdot d\vec{S} = 2\iiint_E dV =$$

$y = y$

$x = r\cos\theta$

$z = r\sin\theta$

$dV = r\,dy\,dr\,d\theta$

$0 \le y \le 2-x$

$0 \le y \le 2-r\cos\theta$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$= 2\int_0^{2\pi}\int_0^1\int_0^{2-r\cos\theta} r\,dy\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^1 (2-r\cos\theta)\,r\,dr\,d\theta$$

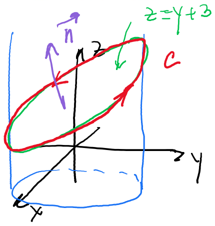
$$= 2\int_0^{2\pi}\int_0^1 (2r - r^2\cos\theta)\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\left(r^2 - \frac{r^3}{3}\cos\theta\right)\Big|_0^1 d\theta$$

$$= 2\int_0^{2\pi}\left(1 - \frac{1}{3}\cos\theta\right) d\theta$$

$$= 2\left(\theta - \frac{1}{3}\sin\theta\right)\Big|_0^{2\pi} = \boxed{4\pi}$$

17. Use Stokes' Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y,z) = <3z, 5x, -2y>$ and $C$ is the ellipse in which the plane $z = y+3$ intersects the cylinder $x^2+y^2=4$, with positive orientation as viewed from above.




$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3z & 5x & -2y \end{vmatrix} = \vec{i}\left(\frac{\partial}{\partial y}(-2y) - \frac{\partial}{\partial z}(5x)\right) - \vec{j}\left(\frac{\partial}{\partial x}(-2y) - \frac{\partial}{\partial z}(3z)\right)$$

$$+\vec{k}\left(\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(3z)\right)$$

$$= -2\vec{i} + 3\vec{j} + 5\vec{k}$$

S: $z = y+3$ ($\vec{n}$ is directed upward)

$$\vec{n} = \pm < z_x, z_y, -1>$$

$$= <-z_x, -z_y, 1>$$

$$= <0, -1, 1>$$

$$\operatorname{curl}\vec{F}\cdot\vec{n} = <-2, 3, 5>\cdot<0,-1,1> = -3+5 = 2$$

---

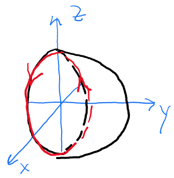
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{x^2+y^2\le 4} 2\, dA$$

$$= 2\iint_{x^2+y^2\le 4} dA$$

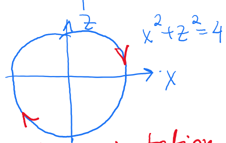
the area of the disk of radius 2

$$= 2\cdot\pi\cdot 2^2 = \boxed{8\pi}$$

18. Use Stokes' Theorem to evaluate $\iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S}$ for the vector field $\mathbf{F}(x, y, z) = \langle ze^y, x\cos y, xz\sin y\rangle$ and the hemisphere $y = \sqrt{4 - x^2 - z^2}$ oriented in the direction of the positive $y$-axis.




projection onto xz plane




the orientation is clock wise

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$$

C: $x^2+z^2=4$ with clockwise orientation.

parametrization

$x = 2\cos t$

$z = -2\sin t$

$0 \le t \le 2\pi$

$\vec{r}(t) = \langle 2\cos t, 0, -2\sin t\rangle$

$\vec{r}\,'(t) = \langle -2\sin t, 0, -2\cos t\rangle$

$$\vec{F}(x,y,z) = \langle ze^y, x\cos y, xz\sin y\rangle \xrightarrow[\substack{x=2\cos t\\ z=-2\sin t}]{y=0} \langle -2\sin t\, e^0, 2\cos t\cos 0, -4\cos t\sin t\,\sin 0\rangle$$

$$= \langle -2\sin t, 2\cos t, 0\rangle$$

$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0\rangle\cdot\langle -2\sin t, 0, -2\cos t\rangle$$

$$= 4\sin^2 t$$

---

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} 4\sin^2 t\, dt$$

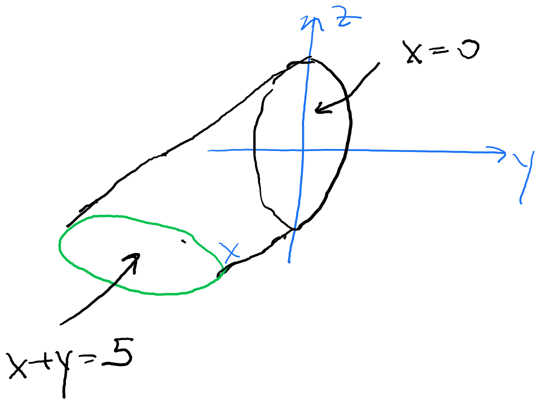
$$= \frac{4}{2}\int_0^{2\pi} (1-\cos 2t)\,dt$$

$$= 2\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi}$$

$$= 2(2\pi) = \boxed{4\pi}$$

19. Use the Divergence Theorem to find the flux of the vector field $\mathbf{F} = \langle x, y, 1 \rangle$ across the surface $S$ which is the boundary of the region enclosed by the cylinder $y^2 + z^2 = 1$ and the planes $x = 0$ and $x + y = 5$.

$$\oiint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$




$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(1) = 2$$

$0 \le x \le 5 - y$

$y = r\cos\theta$

$z = r\sin\theta$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$\oiint_S \vec{F} \cdot d\vec{S} = \iiint_V \operatorname{div} \vec{F} \, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{5 - r\cos\theta} 2 \, r \, dx \, dr \, d\theta$$

$$= 2 \int_0^{2\pi} \int_0^1 (5 - r\cos\theta) \, r \, dr \, d\theta$$

$$= 2 \int_0^{2\pi} \int_0^1 (5r - r^2\cos\theta) \, dr \, d\theta$$

$2\pi$

$$= 2\int_0^{2\pi} \left( \frac{5r^2}{2} - \frac{r^3}{3}\cos\theta \right)\Big|_0^1 \, d\theta$$

$$= 2\int_0^{2\pi} \left( \frac{5}{2} - \frac{1}{3}\cos\theta \right) d\theta$$

$$= \Big[ 5(\theta) - \frac{2}{3}\sin\theta \Big]_0^{2\pi}$$

$$= \boxed{10\pi}$$