Session 5: Sections 2-5 and 2-6 (Implicit Differentiation)

Specific Formulas for the Chain Rule

If g is a differentiable function, n is any real number, and b is any positive real number, then

- **Generalized Power Rule:** If $y = (g(x))^n$, then $y\,' = n(g(x))^{n-1} \cdot g\,'(x)$.
- **Generalized Exponential (base b) Rule:** If $y = b^{g(x)}$, then $y\,' = b^{g(x)}(\ln(b)) \cdot g\,'(x)$.
- **Generalized Logarithm (base b) Rule:** If $y = \log_b(g(x))$, then $y\,' = \dfrac{g\,'(x)}{(g(x))\,(\ln(b))}$.

1. Differentiate each of the following.

(a) $f(x) = \left(5x^4 + 3x^2 + x\right)^{20}$

$$f'(x) = 20\left(5x^4+3x^2+x\right)^{19}\frac{d}{dx}\left(5x^4+3x^2+x\right)$$

$$\boxed{= 20\left(5x^4+3x^2+x\right)^{19}\left(20x^3+6x+1\right)}$$

(b) $g(x) = e^{\sqrt{x}+x^2-5}$

$$g'(x) = \left(e^{x^{1/2}+x^2-5}\right)\underbrace{\left(\ln e\right)}_{=1}\frac{d}{dx}\left(x^{1/2}+x^2-5\right)$$

$$\boxed{= \left(e^{x^{1/2}+x^2-5}\right)\left(\frac{1}{2}x^{-1/2}+2x\right)}$$

(c) $t(x) = \log_5(\underbrace{2x^3 + e^x}_{\text{arguement}})$

$$t'(x) = \frac{\frac{d}{dx}\left(2x^3+e^x\right)}{\left(2x^3+e^x\right)\ln 5} = \boxed{\frac{6x^2+e^x}{\left(2x^3+e^x\right)\ln 5}}$$

(d) $f(x) = e^{e^{2x}}$

$$f'(x) = e^{e^{2x}}\underbrace{\left(\ln e\right)}_{=1}\frac{d}{dx}\left(e^{2x}\right)$$

$$= \underbrace{e^{e^{2x}} \cdot e^{2x}}_{\substack{\text{multiply like} \\ \text{base, add exponents}}} \cdot 2 = \boxed{2e^{e^{2x}+2x}}$$

(e) $y(x) = \underbrace{x^5 2^{x^2+5x}}_{\text{product rule}} + e^{\pi^2}$ ← Constant

$$y'(x) = x^5 \frac{d}{dx}\left(2^{x^2+5x}\right) + 2^{x^2+5x}\frac{d}{dx}\left(x^5\right) + \frac{d}{dx}\left(e^{\pi^2}\right)$$

$$\boxed{= x^5\left(2^{x^2+5x}\right)(2x+5) + 2^{x^2+5}\left(5x^4\right) + 0}$$

(f) $p(x) = \ln\left(\dfrac{2x^2+x}{x+1}\right)$ use property $\log_b\left(\frac{n}{m}\right) = \log_b n - \log_b m$

$$p(x) = \ln\left(2x^2+x\right) - \ln(x+1)$$

$$p'(x) = \frac{\frac{d}{dx}\left(2x^2+x\right)}{\left(2x^2+x\right)\ln e} - \frac{\frac{d}{dx}(x+1)}{(x+1)\ln e}$$

$$\boxed{= \frac{4x+1}{2x^2+x} - \frac{1}{x+1}}$$

(g) $f(x) = \dfrac{\log_5(3x^4+7x)}{\sqrt{2x^3}} = \dfrac{\log_5(3x^4+7x)}{(2x^3)^{1/2}}$

Quotient Rule

$$f'(x) = \frac{(2x^3)^{1/2}\frac{d}{dx}\left(\log_5(3x^4+7x)\right) - \log_5(3x^4+7x)\frac{d}{dx}(2x^3)^{1/2}}{\left((2x^3)^{1/2}\right)^2}$$

$$\boxed{= \frac{(2x^3)^{1/2}\left(\frac{12x^3+7}{(3x^4+7x)\ln 5}\right) - \log_5(3x^4+7x)\left(\frac{1}{2}(2x^3)^{-1/2}(6x^2)\right)}{2x^3}}$$

(h) $p(x) = \left[\log_2\left(x^2+e^{3x^2}\right)\right]^4$

$$\boxed{p'(x) = 4\left(\log_2\left(x^2+e^{3x^2}\right)\right)^3\left(\frac{\overbrace{2x+e^{3x^2}\cdot 6x\cdot\ln e}^{d/dx\left(x^2+e^{3x^2}\right)}}{\left(x^2+e^{3x^2}\right)\ln 2}\right)}$$

2. Find the $x-$values where the graph of $f(x) = x^3 e^{-x^2}$ has horizontal tangent lines.

$$f'(x) = x^3 \frac{d}{dx} e^{-x^2} + \frac{d}{dx} x^3 \cdot e^{-x^2}$$

$$= x^3 \left(e^{-x^2}\right)(-2x) + 3x^2 e^{-x^2}$$

$$= x^2 e^{-x^2} \left(-2x^2 + 3\right)$$

$$0 = x^2 e^{-x^2} (-2x^2 + 3)$$

$x^2 = 0$ $\quad$ $\cancel{e^{-x^2} = 0}$ $\quad$ $-2x^2 + 3 = 0$

$\boxed{x=0}$ $\quad$ Never $\quad$ $-2x^2 = -3$

$x^2 = 3/2$

$\boxed{x = \pm\sqrt{\frac{3}{2}}}$

3. Suppose $F(x) = g(f(x))$ and $f(2) = 3, f'(2) = -3, g(3) = 5$, and $g'(3) = 4$. Find $F'(2)$.

$$F'(x) = g'(f(x)) f'(x)$$

$$F'(2) = g'(\underbrace{f(2)}_{3}) \underbrace{f'(2)}_{-3}$$

$$= \underbrace{g'(3)}_{4}(-3)$$

$$= 4(-3)$$

$$= \boxed{-12}$$

4. Determine an equation for the line tangent to $f(x) = \ln(3x)$ at $x = 2$.

$f'(x) = \frac{3}{3x} = \frac{1}{x}$ $f'(2) = \frac{1}{2}$ $f(2) = \ln(6)$

$y - \ln(6) = \frac{1}{2}(x-2)$

$\boxed{y = \frac{1}{2}x - 1 + \ln 6}$

Steps for using implicit differentiation to find $\frac{dy}{dx}$.

Step 1: Substitute $y(x)$ because we are assuming y is a function of x.

Step 2: Take the derivative of both sides of the equation with respect to x.

Step 3: Substitute y back in for $y(x)$ and write $y\,'(x)$ as $\frac{dy}{dx}$.

Step 4: Solve for $\frac{dy}{dx}$.

5. Use implicit differentiation to find $\frac{dy}{dx}$ for each of the following.

(a) $-6e^y + \frac{1}{3}y^3 - 2\ln(x) = -8$

$-6e^{y(x)} + \frac{1}{3}(y(x))^3 - 2\ln(x) = -8$

$-6e^{y(x)} y'(x) \ln e + (y(x))^2 y'(x) - \frac{2}{x} = 0$

$-6e^y \frac{dy}{dx} + y^2 \frac{dy}{dx} = \frac{2}{x}$

$\frac{dy}{dx}(-6e^y + y^2) = \frac{2}{x} \longrightarrow \frac{dy}{dx} = \frac{2}{x} \cdot \frac{1}{-6e^y + y^2} = \boxed{\frac{2}{x(-6e^y + y^2)}}$

(b) $x^2 + x + y^3 - 3y^2 = 45$

$x^2 + x + (y(x))^3 - 3(y(x))^2 = 45$

$2x + 1 + 3(y(x))^2 y'(x) - 6y(x)y'(x) = 0$

$2x + 1 + 3y^2 \frac{dy}{dx} - 6y\frac{dy}{dx} = 0$

$3y^2 \frac{dy}{dx} - 6y\frac{dy}{dx} = -2x - 1$

$\frac{dy}{dx}(3y^2 - 6y) = -2x - 1$

$\boxed{\frac{dy}{dx} = \frac{2x-1}{3y^2 - 6y}}$

product rule

(c) $x^2 + xy^3 - 3y^2 = 45$

$$x^2 + \underbrace{x\,(y(x))^3}_{\text{product rule}} - 3\,(y(x))^2 = 45$$

$$2x + x\,\frac{d}{dx}\left(y(x)\right)^3 + (y(x))^3\,\frac{d}{dx}(x) - 6\,y(x)\,y'(x) = 0$$

$$2x + x \cdot 3\,(y(x))^2\,y'(x) + (y(x))^3\,(1) - 6\,y(x)\,y'(x) = 0$$

$$2x + x \cdot 3y^2\,\frac{dy}{dx} + y^3 - 6y\,\frac{dy}{dx} = 0$$

$$3xy^2\,\frac{dy}{dx} - 6y\,\frac{dy}{dx} = -2x - y^3$$

$$\frac{dy}{dx}\left(3xy^2 - 6y\right) = -2x - y^3$$

$$\boxed{\frac{dy}{dx} = \frac{-2x - y^3}{3xy^2 - 6y}}$$

(d) $3x^4 - \ln\left(y^2\right) = \dfrac{4}{y^2}$

$$3x^4 - 2\ln y = 4y^{-2}$$

$$3x^4 - 2\ln(y(x)) = 4\,(y(x))^{-2}$$

$$12x^3 - 2\,\frac{y'(x)}{y(x)\ln e} = -8\,(y(x))^{-3}\,y'(x)$$

$$12x^3 - \frac{2\,dy/dx}{y} = -8\,y^{-3}\,dy/dx$$

$$12x^3 = -8y^{-3}\,dy/dx + \frac{2}{y}\,dy/dx$$

$$12x^3 = \left(-8y^{-3} + \frac{2}{y}\right)\frac{dy}{dx}$$

$$\boxed{\frac{12x^3}{-8y^{-3} + \frac{2}{y}} = \frac{dy}{dx}}$$

6. Find the slope of the line tangent to curve given by $2x^3 - 10\log_3(y^4) = \dfrac{120}{y} + 48$ at the point $(4, 3)$. If necessary, round to the nearest hundredth.

$$2x^3 - 10\log_3\left(y(x)\right)^4 = 120\left(y(x)\right)^{-1} + 48$$
$$2x^3 - 40\log_3\left(y(x)\right) = 120\left(y(x)\right)^{-1} + 48$$
$$6x^2 - 40\left(\frac{y'(x)}{y(x)\ln 3}\right) = -120\left(y(x)\right)^{-2} y'(x)$$
$$6x^2 - \frac{40\, dy/dx}{y\ln 3} = -120y^{-2}\frac{dy}{dx}$$
$$6x^2 = \frac{40}{y\ln 3}\frac{dy}{dx} - 120y^{-2}\frac{dy}{dx}$$
$$6x^2 = \left(\frac{40}{y\ln 3} - 120y^{-2}\right)\frac{dy}{dx}$$
$$\frac{6x^2}{\frac{40}{y\ln 3} - \frac{120}{y^2}} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6\cdot 4^2}{\frac{40}{3\ln 3} - \frac{120}{3^2}}$$
$$\approx -80.21$$
$$y - 3 = -80.21(x-4)$$
$$y = -80.21x + 320.84 + 3$$
$$\boxed{y = -80.21x + 323.84}$$

7. Suppose x and y are functions of t and are related by the equation $\ln\left(3x^2 - 5y\right) + 16 = x^4$. Given $\dfrac{dx}{dt} = 3$, $x = 2$, and $y = \dfrac{11}{5}$, use the information to find $\dfrac{dy}{dt}$.

$$\ln\left(3\left(x(t)\right)^2 - 5y(t)\right) + 16 = \left(x(t)\right)^4$$
$$\frac{6\left(x(t)\right)x'(t) - 5y'(t)}{3\left(x(t)\right)^2 - 5y(t)} = 4\left(x(t)\right)^3 x'(t)$$
$$\frac{6x\frac{dx}{dt} - 5\frac{dy}{dt}}{3x^2 - 5y} = 4x^3\frac{dx}{dt}$$
$$\frac{6(2)(3) - 5\frac{dy}{dt}}{3(2)^2 - 5\left(11/5\right)} = 4(2)^3(3)$$

$$\frac{36 - 5\frac{dy}{dt}}{1} = 96$$
$$-5\frac{dy}{dt} = 96 - 36 = 60$$
$$\boxed{\frac{dy}{dt} = \frac{60}{-5} = -12}$$