Math 152 - Week-in-Review 9

$\begin{cases} \ln(a)+\ln(b)=\ln(ab) \\ \ln(a+b) \text{ is irreducible} \end{cases}$

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Find the a) Radius and the b) Interval of Convergence for the following power series.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{3^n(x-1)^n}{n\ln(n)}$.

$\longrightarrow$ a : $(x-1)=0$
$x=1 \leftarrow$ center

x^n or $(x-a)^n$
$\downarrow$
Ratio Test
$\downarrow$
$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ for convergence.

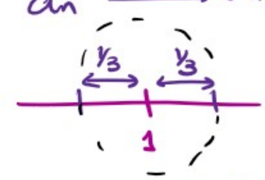
$a_n \xrightarrow{n\to(n+1)} a_{n+1}$

RT: $\displaystyle\lim_{n\to\infty}\left|\frac{\cancel{3^{n+1}}^{\,3}\,\cancel{(x-1)^{n+1}}^{\,(x-1)}}{(n+1)\cdot\ln(n+1)}\cdot\frac{n\ln(n)}{\cancel{3^n}\cdot\cancel{(x-1)^n}}\right|$

$\displaystyle |3(x-1)|\left[\lim_{n\to\infty}\frac{n\ln(n)}{(n+1)\ln(n+1)}\right] \to \lim_{n\to\infty}\frac{n\ln(n)}{(n)\ln(n)} = 1$

$|3(x-1)| < 1$ for convergence.

$|(x-1)| < \frac{1}{3} \Rightarrow R=\frac{1}{3}$ a)



$-1 < 3(x-1) < 1$
$-\frac{1}{3} < x-1 < \frac{1}{3}$
$\boxed{\frac{2}{3} < x < \frac{4}{3}}$

$R=\left(\frac{4}{3}-\frac{2}{3}\right)/2 = \frac{1}{3}$

b) $\left[\frac{2}{3}, \frac{4}{3}\right)$

$(x-1) = \left(\frac{2}{3}-1\right) = \left(-\frac{1}{3}\right)$

Test $x=2/3$: $\displaystyle\sum_{n=2}^{\infty}\frac{3^n(x-1)^n}{n\ln(n)} = \sum_{n=2}^{\infty}\frac{3^n\cdot\left(-\frac{1}{3}\right)^n}{n\ln(n)}$

$\displaystyle = \sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln(n)} \to$ converges by (AST)

Test $x=4/3$: $\displaystyle\sum_{n=2}^{\infty}\frac{3^n(x-1)^n}{n\ln(n)} = \sum_{n=2}^{\infty}\frac{3^n\left(\frac{1}{3}\right)^n}{n\ln(n)}$

$\displaystyle = \sum_{n=2}^{\infty}\frac{(1)^n}{n\ln(n)} = \sum_{n=2}^{\infty}\frac{1}{n\ln(n)}$

$\longrightarrow$ diverges by (Integral test)

2. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{n!}$.

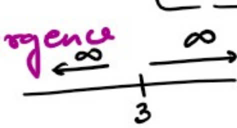
$x-3=0$
$x=3$

RT: $\displaystyle\lim_{n\to\infty}\left|\frac{\cancel{2^{n+1}}^{\,2}\,\cancel{(x-3)^{n+1}}^{\,(x-3)}}{\cancel{(n+1)!}_{\,(n+1)}}\cdot\frac{\cancel{n!}}{\cancel{2^n}\,\cancel{(x-3)^n}}\right|$

$n!(n+1) = (n+1)!$

$\displaystyle = |2(x-3)|\cdot\lim_{n\to\infty}\left(\frac{1}{n+1}\right)^{\!0} = 0$

means convergence

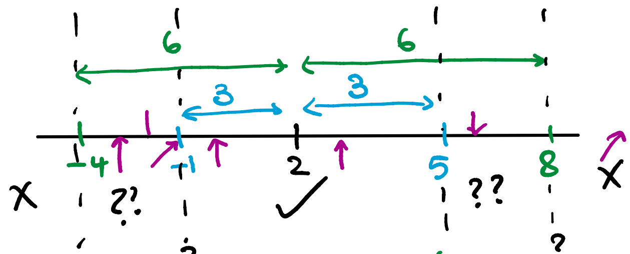


a) $R=\infty$

b) Interval of convergence: $(-\infty,\infty)$
$\Downarrow$
all values of x

center: $a=2$

3. If the power series given by $\sum_{n=0}^{\infty} C_n(x-2)^n$ coverges at $x=5$ and diverges at $x=-4$, what can we say about the following?



$3 < R < 6$

IC: $(-1, 5]$ or $[-1, 5]$

IC: $(-4, 8)$ or $(-4, 8]$

(a) $\sum_{n=0}^{\infty} C_n$ $= \sum_{n=0}^{\infty} C_n (1)^n$

$x-2=1$
$x=3$ } series converges

(b) $\sum_{n=0}^{\infty} C_n(-3)^n$ $= \sum_{n=0}^{\infty} C_n (x-2)^n$

$x-2=-3$
$x=-1$ } unknown

(c) $\sum_{n=0}^{\infty} C_n 9^n$

$x-2=9$
$x=11$
series diverges

(d) $\sum_{n=0}^{\infty} C_n(-5)^n$

$x-2=-5$
$x=-3$
unknown

(e) $\sum_{n=0}^{\infty} C_n(-2)^n$

$x-2=-2$
$x=0$
series converges

(f) $\sum_{n=0}^{\infty} C_n 4^n$

$x-2=4$
$x=6$
unknown

$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n = \underset{n=0}{C_0} + \underset{n=1}{C_1(x-a)} + \underset{n=2}{C_2(x-a)^2} + \cdots$

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Math 152 - Fall 2024
WIR 8: 11.8 – 11.10

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = \overset{n=0}{c_0} + \overset{n=1}{c_1(x-a)} + \overset{n=2}{c_2(x-a)^2} + \cdots$$

Building blocks.

Let's recap the Power Series essentials: Maclaurin series $(a=0)$

① $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $|x|<1$ or $R=1$ $(-1,1)$

② $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $R \to \infty$

③ $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$ $R \to \infty$

④ $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$ $R \to \infty$

⑤ $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)}$ ← not factorial $R=1$

⑥ $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{n+1}}{(n+1)}$

Find a Power Series representation for the following functions. Give the radius and interval of convergence.

4. $f(x) = \frac{x}{25x^2 - 4}$ $= x \cdot \left(\frac{1}{25x^2-4}\right) = x \cdot g(x) = f(x)$

$$g(x) = \frac{1}{25x^2-4} = \frac{1}{-4+25x^2} = \frac{1}{-4\left(1-\frac{25x^2}{4}\right)}$$

$$g(x) = \left(-\frac{1}{4}\right)\left(\frac{1}{1-\frac{25x^2}{4}}\right) = \left(-\frac{1}{4}\right)\sum_{n=0}^{\infty}\left(\frac{25x^2}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1) \cdot 25^n \cdot x^{2n}}{4 \cdot 4^n} = \sum_{n=0}^{\infty} \frac{(-1) \cdot 25^n \cdot x^{2n}}{4^{n+1}} = g(x)$$

$$f(x) = x \cdot g(x) = x \sum_{n=0}^{\infty} \frac{(-1)\, 25^n}{4^{n+1}} \cdot x^{2n}$$

$4 \cdot 4^n = 4^1 \cdot 4^n = 4^{n+1}$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1) \cdot 25^n \cdot x^{2n+1}}{4^{n+1}}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

5. $f(x) = \dfrac{1}{(2-x)^2}$

$$\int f(x)dx = \int \frac{1}{(2-x)^2}dx = \int -\frac{du}{u^2}$$

$u = 2-x$
$du = -dx$

$$= -\left(-\frac{1}{u}\right)$$

$$= +\frac{1}{2-x}$$

$$\int f(x)dx = \frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{x=0}^{\infty}\frac{x^n}{2^{n+1}}$$

$$f(x) = \frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$$

$$= \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\cdot\underbrace{\frac{d}{dx}(x^n)}$$

$$f(x) = \frac{1}{(2-x)^2} = \sum_{n=1}^{\infty}\frac{1}{2^{n+1}}\cdot n\,x^{n-1} \;\underset{\boxed{n\to(n+1)}}{=}\; \sum_{n=0}^{\infty}\frac{1}{2^{n+2}}\cdot(n+1)\cdot x^n$$

$$\frac{1}{2^2}\cdot = \frac{1}{4} \qquad \frac{1}{2^2}(0+1)\cdot x^0 = \frac{1}{4}$$

6. $f(x) = \dfrac{x^2}{(2-x)^2}$

$$f(x) = (x^2)\,\frac{1}{(2-x)^2} = x^2\cdot\sum_{n=0}^{\infty}\frac{1}{2^{n+2}}(n+1)x^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{2^{n+2}}(n+1)\cdot x^{n+2}$$

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{(n+1)}$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)}$$

7. $\boxed{f(x) = \ln(1+5x^2)}$ 1

$$f'(x) = \left(\frac{1}{1+5x^2}\right)(10x) = (10x)g(x)$$

2

$$g(x) = \frac{1}{1+5x^2} = \frac{1}{1-(-5x^2)} \sim \frac{1}{1-*}$$

$$= \sum_{n=0}^{\infty} (-5x^2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot 5^n \cdot x^{2n}$$

undo 2

$$f'(x) = (10x) \sum_{n=0}^{\infty} (-1)^n \cdot 5^n \cdot x^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n\, 10 \cdot (5^n)\, x^{2n+1}$$

undo 1

$$f(x) = \int f'(x)\,dx = \sum_{n=0}^{\infty} (-1)^n \cdot 10 \cdot (5^n) \int x^{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot (10)\,(5^n) \cdot \frac{x^{2n+2}}{2n+2} + C$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{\cancel{10}^{\,5}\,(5^n) \cdot x^{2n+2}}{\cancel{2}(n+1)} + C$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{5^{n+1} \cdot x^{2n+2}}{(n+1)} + C \;?$$

Solving for C → use x=0

PS = 0.

$$\ln(1+5x^2) \xrightarrow{x=0} \ln(1) = 0$$

$\therefore C = 0$.

Alternate method

Using Maclaurin series for $\ln(1+x)$

$$\ln(1+5x^2)$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(5x^2)^{n+1}}{(n+1)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{5^{n+1} \cdot x^{2n+2}}{(n+1)}$$

8. $f(x) = \arctan(2x)$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)}$$

$$\arctan(2x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(2x)^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{2^{2n+1} \cdot x^{2n+1}}{(2n+1)}$$

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Math 152 - Fall 2024
WIR 8: 11.8 – 11.10

Evaluate the following integrals as Power Series.

9. $f(x) = \int \frac{1}{1+x^4}\, dx \quad = \int g(x)\,dx$

$$g(x) = \frac{1}{1+x^4} = \frac{1}{1-(-x^4)} = \sum_{n=0}^{\infty} (-x^4)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot x^{4n}$$

$$f(x) = \int \sum_{n=0}^{\infty} (-1)^n \cdot x^{4n}\,dx = \sum_{n=0}^{\infty} (-1)^n \int x^{4n}\,dx$$

$$\int \frac{1}{1+x^4}\,dx = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{4n+1}}{4n+1} + C$$

here C stays

10. $f(x) = \int x^2 \arctan(3x^2)\, dx \quad = \int x^2 \cdot g(x)\,dx$

① $g(x) = \arctan(3x^2) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(3x^2)^{2n+1}}{(2n+1)}$

$(x^2)^{2n+1}$
$2(2n+1) = 4n+2$

$4n+2+2 = 4n+4$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{2n+1} \cdot x^{4n+2}}{(2n+1)}$$

② $x^2 g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{2n+1}}{(2n+1)} \cdot x^{4n+4}$

③ $f(x) = \int x^2 g(x)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{2n+1}}{(2n+1)} \cdot \int x^{4n+4}\,dx$

$$f(x) = \int x^2 \arctan(3x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{2n+1}}{(2n+1)} \cdot \frac{x^{4n+5}}{(4n+5)} + C$$

11. If $f(x) = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$, find the power series for $f'(x)$ and $\int f(x)dx$. Identify $f(x)$.

12. Find the 25^{th} derivative for the function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+2)} x^n$ centered at $x = 0$.

Find the Taylor Series Representions for the following functions

13. $f(x) = e^{3x}$ centered at $x = 5$

14. $f(x) = \ln(x)$ centered at $a = 2$

Find the Maclaurin Series Representation for the following functions.

15. $f(x) = \displaystyle\int_0^x e^{-t^2}\, dt.$

16. $f(x) = x^3 \cos(x)$

Find the sum of the following series.

17. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(3)^n(\pi^n)}{n!}$

18. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!}$

19. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n)!}$

20. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{2^{2n+1}(2n)!}$

$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

a) TOD:
$\lim a_n = \lim \frac{1}{n\ln(n)} = \frac{1}{\infty} = 0.$

$$\sum_{n=2} a_n = \sum_{n=2} \frac{1}{n\ln(n)}$$

$a_n = \frac{1}{n\ln(n)}$, n = integer

$f(x) = \frac{1}{x\ln(x)} \rightarrow$ (+)ve, continuous, decreasing on $[2,\infty)$

a) TOD:

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n\ln(n)} = \frac{1}{\infty} = 0.$

Test fails.

b) Integral test.

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x\ln(x)}\,dx$$

$u = \ln(x)$
$du = \frac{1}{x}dx$
$\longrightarrow \int \frac{du}{u} = \ln|u|$?

Formula:

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\int \frac{1}{x}dx = \ln|x|$

$$= \ln|\ln(x)|\Big|_{x=2}^{x=\infty}$$

$$= \underbrace{\ln|\ln(\infty)|}_{\to\infty} - \ln|\ln(2)|$$

$\therefore$ Integral diverges

Since $\int f(x)dx$ diverges, then $\sum a_n$ also diverges.

CalcFormula

Spring 2019 Math 152
Formulas from Calculus I
courtesy: Amy Austin

Derivatives

1. $\frac{d}{dx}x^n = nx^{n-1}$

2. $\frac{d}{dx}\ln x = \frac{1}{x}$

3. $\frac{d}{dx}\ln(g(x)) = \frac{g'(x)}{g(x)}$

4. $\frac{d}{dx}e^x = e^x$

5. $\frac{d}{dx}a^x = a^x \ln a$

6. $\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}$

7. $\frac{d}{dx}a^{g(x)} = g'(x)a^{g(x)}\ln a$

8. $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

9. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

10. $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

11. $\frac{d}{dx}\sin x = \cos x$

12. $\frac{d}{dx}\cos x = -\sin x$

13. $\frac{d}{dx}\tan x = \sec^2 x$

14. $\frac{d}{dx}\sec x = \sec x \tan x$

15. $\frac{d}{dx}\csc x = -\csc x \cot x$

16. $\frac{d}{dx}\cot x = -\csc^2 x$

17. Product Rule: $\frac{d}{dx}gh = g'h + gh'$

18. Quotient Rule: $\frac{d}{dx}\frac{g}{h} = \frac{g'h - gh'}{h^2}$

19. Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

Integrals

20. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, if $n \neq -1$

21. $\int e^x\,dx = e^x + C$

22. $\int a^x\,dx = \frac{a^x}{\ln a} + C$

23. $\int \frac{1}{x}\,dx = \ln|x| + C$

24. $\int \frac{1}{1+x^2}\,dx = \arctan x + C$

25. $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

26. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

27. $\int \cos x\,dx = \sin x + C$

28. $\int \sin x\,dx = -\cos x + C$

29. $\int \sec x \tan x\,dx = \sec x + C$

30. $\int \sec^2 x\,dx = \tan x + C$

31. $\int \csc x \cot x\,dx = -\csc x + C$

32. $\int \csc^2 x\,dx = -\cot x + C$

Logarithm Rules

33. $\ln PQ = \ln P + \ln Q$

34. $\ln \frac{P}{Q} = \ln P - \ln Q$

35. $\ln P^r = r \ln P$

Useful Trig Identities

36. $\cos^2 x + \sin^2 x = 1$

37. $\tan^2 x + 1 = \sec^2 x$

38. $\cos^2 x = \frac{1}{2}[1 + \cos 2x]$

39. $\sin^2 x = \frac{1}{2}[1 - \cos 2x]$

40. $\sin 2x = 2\sin x \cos x$

41. $\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$