IBP → 2 types of fn ⇒ $\int u\,dv = uv - \int v\,du$

I L A T E

Math 152 - Week-In-Review 4 (Exam 1 review)

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1. Evaluate the indefinite integral $\int x^3 \ln x\, dx$. → IBP ⇒ $(x^3)(\ln x)$

$u = \ln(x)$ $\quad dv = x^3 dx$

$du = \frac{1}{x} dx$ $\quad v = \int x^3 dx = \frac{x^4}{4}$

$$\int x^3 \ln(x)\,dx = \underbrace{\ln(x)\cdot\left(\frac{x^4}{4}\right)}_{uv} - \underbrace{\int \left(\frac{x^4}{4}\right)\left(\frac{1}{x}dx\right)}_{\int v\,du}$$

$$= \frac{1}{4}x^4 \ln(x) - \frac{1}{4}\int x^3 dx$$

$$= \frac{1}{4}x^4 \ln(x) - \frac{1}{4}\left(\frac{x^4}{4}\right) + C$$

$$\int x^3 \ln(x)\,dx = \frac{1}{4}x^4 \ln(x) - \frac{1}{16}\cdot x^4 + C$$

2. Evaluate the definite integral $\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx$. → IBP

$1\,dx$

$x^0\,dx$

$u = \arctan\left(\frac{1}{x}\right)$ $\quad dv = dx$

$du = \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\frac{d}{dx}\left(\frac{1}{x}\right)\cdot dx$ $\quad v = \int dx = x$

$$= \frac{1}{\left(1+\frac{1}{x^2}\right)}\cdot\left(-\frac{1}{x^2}\right)dx$$

$$= \frac{1}{\left(\frac{x^2+1}{x^2}\right)}\cdot\left(-\frac{1}{x^2}\right)dx$$

$$= \frac{\cancel{x^2}}{x^2+1}\cdot\left(-\frac{1}{\cancel{x^2}}\right)dx$$

$$du = \frac{-1}{1+x^2}dx$$

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)dx = x\arctan\left(\frac{1}{x}\right) - \int x\cdot\left(\frac{-1}{1+x^2}\right)dx$$

$$- \int \frac{x}{1+x^2}dx$$

$t = 1+x^2$

$dt = 2x\,dx$

$$= -\frac{1}{2}\int\frac{dt}{t}$$

$$= -\frac{1}{2}\ln|t|$$

$$= x\arctan\left(\frac{1}{x}\right) + \frac{1}{2}\ln(1+x^2)\Big|_{x=1}^{x=\sqrt{3}}$$

$$= \left[\sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{2}\ln(1+3)\right] - \left[1\arctan(1) + \frac{1}{2}\ln(1+1)\right]$$

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)dx = \sqrt{3}\cdot\frac{\pi}{6} + \frac{1}{\cancel{2}}\overbrace{\cancel{\ln}(4)}^{\ln(2)} - \frac{\pi}{4} - \frac{1}{2}\ln(2) = \sqrt{3}\frac{\pi}{6} - \frac{\pi}{4} + \frac{1}{2}\ln(2)$$

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)dx = \sqrt{3}\cdot\frac{\pi}{6} + \frac{1}{\cancel{2}}\cancel{\ln}(4) - \frac{\pi}{4} - \frac{1}{2}\ln(2) = \sqrt{3}\frac{\pi}{6} - \frac{\pi}{4} + \frac{1}{2}\ln(2)$$

3. Evaluate the indefinite integral $\int x^2 e^x \, dx$

$u = x^2 \qquad dv = e^x dx$

$du = 2x\,dx \qquad v = e^x$

$$\int x^2 \cdot e^x\,dx = x^2\cdot e^x - \underbrace{\int e^x \cdot (2x\,dx)}$$

TABULAR METHOD

u		v
x^2	$+$	e^x
$2x$	$(-)$	e^x
2	$+$	e^x
0		e^x

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + C$$

4. Evaluate the definite integral $\int_{x=0}^{x=1} \frac{x}{\sqrt{1+x^2}}\,dx$

$u = 1+x^2$, $du = 2x\,dx$, $\frac{du}{2} = x\,dx$; $x=1$: $u=2$; $x=0$: $u=1$

$$= \frac{1}{2}\int_{u=1}^{u=2} \frac{du}{\sqrt{u}} = \frac{1}{2}\int_1^2 u^{-1/2}\,du$$

$$= \frac{1}{2}\,\frac{u^{1/2}}{1/2}\Big|_1^2 = u^{1/2}\Big|_1^2$$

$$= 2^{1/2} - 1$$

5. Evaluate the definite integral $\int_0^{\pi} e^{\cos t}\sin 2t\,dt$.

$x = \cos t$, $dx = -\sin t\,dt$; $t=\pi$: $x=-1$; $t=0$: $x=1$

$\boxed{\sin(2t) = 2\sin t\cos t}$

$$2\int_0^{\pi} e^{\cos t}\cdot \sin t\cos t\,dt$$

$$= 2\int_1^{-1} e^x \cdot x\,(-dx)$$

$$= -2\int_1^{-1} xe^x\,dx = 2\int_{-1}^{1} xe^x\,dx$$

u		v
x	$+$	e^x
1	$-$	e^x
0		e^x

$$= 2\left[\,xe^x - e^x\Big|_{-1}^{1}\right] = 2\left[\cancel{e} - \cancel{e} - (\underbrace{-e^{-1} - e^{-1}}_{-2e^{-1}})\right]$$

$= 2(2e^{-1}) = 4e^{-1} = \frac{4}{e}$ Ans.

6. Evaluate the indefinite integral $\int e^{3x} \cos x \, dx$. $\to I$ $\to$ LOOP $\to$ IBP twice

$u = e^{3x}$ $\quad dv = \cos x \, dx$

$du = 3e^{3x} dx$ $\quad v = \sin(x)$

IBP ①

$$I = e^{3x} \cdot \sin(x) - 3\int \sin(x) \cdot e^{3x} dx$$

$u = e^{3x}$ $\quad dv = \sin(x) dx$

$du = 3e^{3x} dx$ $\quad v = -\cos(x)$

IBP ②

$$I = e^{3x} \sin(x) - 3\left[e^{3x} \cdot (-\cos x) - \int -\cos(x) \cdot 3 e^{3x} dx \right]$$

$$I = e^{3x} \sin(x) + \underbrace{3e^{3x}\cos x} - \underbrace{9\int e^{3x} \cos(x) \, dx}$$

I

$$I = e^{3x} \sin(x) + 3e^{3x} \cos(x) - 9I$$

$+9I \qquad +9I$

$$10I = e^{3x} \sin(x) + 3e^{3x} \cos(x)$$

$$I = \int e^{3x} \cos(x) dx = \frac{1}{10}\left[e^{3x} \sin(x) + 3e^{3x} \cos(x) \right] + C$$

7. Evaluate the indefinite integral $\int x^5 \sqrt{x^3 + 1} \, dx$

$u = x^3 + 1 \longrightarrow x^3 = u - 1$

$du = 3x^2 dx$

$\frac{du}{3x^2} = dx$

$$\int x^5 \sqrt{u} \cdot \frac{du}{3x^2} = \frac{1}{3}\int x^3 \sqrt{u} \, du$$

$$= \frac{1}{3}\int (u-1)\sqrt{u} \, du = \frac{1}{3}\int \left(u^{3/2} - u^{1/2}\right) du$$

$$= \frac{1}{3}\left[\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} \right] = \frac{1}{3}\left[\frac{2}{5}(x^3+1)^{5/2} - \frac{2}{3}(x^3+1)^{3/2} \right] + C$$

$\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}\cos x = -\sin x$

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x \longrightarrow \tan^2 x = \sec^2 x - 1$

$\csc^2 x = 1 + \cot^2 x$

$\frac{d}{dx}\tan(x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec(x)\tan(x)$

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8. Evaluate the indefinite integral $\int \sec^5 x \tan^3 x \, dx$.

~~$u = \tan x$~~

$u = \sec x$

$du = \sec(x)\tan(x)dx$

$$\int \sec^5(x)\tan^3(x)\,dx = \int \underbrace{\cancel{\sec^4(x)}}_{u^4} \underbrace{\tan^2(x)}_{u^2-1} \underbrace{\cancel{\sec(x)\tan(x)dx}}_{du}$$

$$= \int u^4(u^2-1)\,du = \int (u^6 - u^4)\,du = \frac{u^7}{7} - \frac{u^5}{5} + C$$

$$= \boxed{\frac{\sec^7(x)}{7} - \frac{\sec^5(x)}{5} + C}$$

New Q.

$$\int(\tan^2 x + \tan^4 x)\,dx = \int \tan^2 x(1+\tan^2 x)\,dx = \int \tan^2 x \sec^2 x\,dx$$

$$= \int u^2\,du = \frac{u^3}{3} \Rightarrow \boxed{\frac{1}{3}\tan^3(x) + C}$$

$u = \tan(x)$

$du = \sec^2 x\,dx$

9. Evaluate the indefinite integral $\int \frac{\sec\theta\tan\theta}{4+\sec\theta}\,d\theta$

$u = 4 + \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$$= \int \frac{du}{u} = \ln|u|$$

$$= \ln|4 + \sec\theta| + C$$

New Q.

$$\int_0^{\pi/4} \tan^4(x)\,dx = \int_0^{\pi/4} \tan^2 x \cdot \tan^2 x\,dx = \int_0^{\pi/4} \tan^2 x(\sec^2 x - 1)\,dx$$

$$= \int_0^{\pi/4} \tan^2 x \sec^2 x\,dx - \int_0^{\pi/4} \tan^2 x\,dx$$

$\sec^2 x - 1$

$-\int_0^{\pi/4} \sec^2 x\,dx + \int_0^{\pi/4} 1\,dx$

$$= \frac{1}{3}\tan^3(x)\Big|_0^{\pi/4} - \tan(x)\Big|_0^{\pi/4} + x\Big|_0^{\pi/4}$$

$$= \frac{1}{3}(1-0) - (1-0) + \left(\frac{\pi}{4} - 0\right) = \frac{1}{3} - 1 + \frac{\pi}{4}$$

10. Evaluate $\int_0^{\pi/4} (\sec^2 x)e^{\tan x}\, dx$.

(a) $e^{\sqrt{2}/2} - 1$
(b) $e^{\sqrt{2}} - 1$
(c) $e^{1/2} - 1$
(d) $1 - e$
(e) $e - 1$

$u = \tan x$
$du = \sec^2 x\, dx$

$$\int e^u\, du = e^u = e^{\tan(x)} \Big|_0^{\pi/4}$$
$$= e^1 - e^0$$
$$= e - 1$$

11. Compute $\int_0^{\pi/4} x \cos x\, dx$.

(a) $\frac{\sqrt{2}}{2}\left(\frac{\pi}{4} + 1\right)$
(b) $\frac{\sqrt{2}}{2}\left(\frac{\pi}{4} + 1\right) - 1$
(c) $\frac{\pi}{4} + \frac{\sqrt{2}}{2}$
(d) $\sqrt{2} - 1$
(e) $\frac{\pi\sqrt{2}}{8}$
(f) 0

u		v
x		$\cos x$
1	$+$	$\sin(x)$
0	$(-)$	$-\cos(x)$

$$x\sin(x) + \cos(x) \Big|_0^{\pi/4}$$
$$= \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right) - [0 + 1]$$
$$= \frac{\sqrt{2}}{2}\left(\frac{\pi}{4} + 1\right) - 1$$

$\sin(2x) = 2\sin(x)\cos(x)$
$\sin(4x) = 2\sin(2x)\cos(2x)$

12. Which of the following is the definite integral $\int_0^{\pi/2} \underbrace{\sin(2x)\cos(2x)}\, dx$ equal to?

(a) 3/2
(b) 2/3
(c) 1/2
(d) 1
(e) 0

$$= \int_0^{\pi/2} \frac{\sin(4x)}{2}\, dx = \frac{1}{2}\left(-\frac{\cos(4x)}{4}\right)\Big|_0^{\pi/2}$$
$$= -\frac{1}{8}\left[\underbrace{\cos\left(4 \cdot \frac{\pi}{2}\right)}_{1} - \underbrace{\cos(4 \cdot 0)}_{1}\right]$$
$$= 0$$

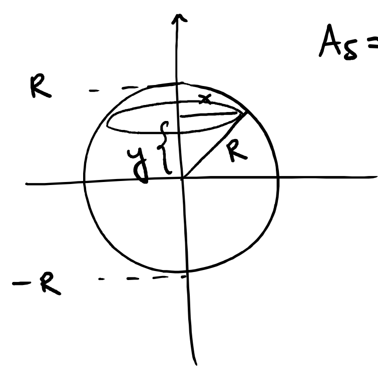
13. A conical tank is 3 feet tall, has a 2 foot radius at the top and is full of water with weight density ρg. The tank has an additional 1 foot spout at the top of the tank. Find the work required to pump all the water out of the spout.

Circle

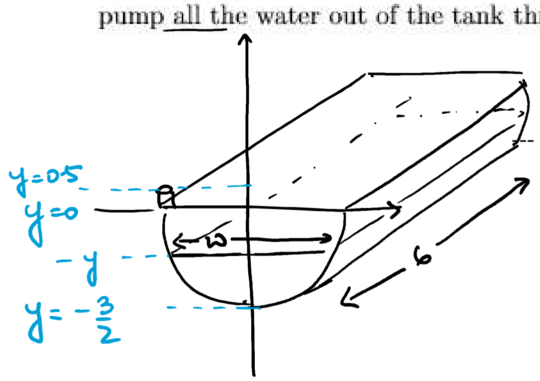
$A_s = \pi x^2$

$x^2 + y^2 = R^2$

$x^2 = R^2 - y^2$



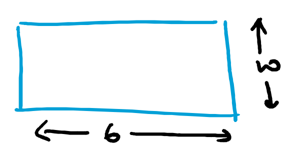
14. A 6 meter long tank with a semi-circular cross section is full of water, with weight denisty $\rho g = 9800$ Newtons per cubic meter. The diameter of the semi-circle is 3 meters. There is a 0.5 meter nozzle at the top of the tank. Find the work required to pump all the water out of the tank through the nozzle.

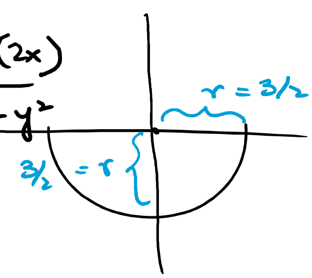


$A_s = 6w = 6(2x)$

$= 6 \cdot 2 \cdot \sqrt{\left(\frac{3}{2}\right)^2 - y^2}$

x-sec





$w = 2x$

$x^2 = R^2 - y^2$

$x = \sqrt{\left(\frac{3}{2}\right)^2 - y^2}$

$$W = \rho g \int_{-3/2}^{0} \left(6 \cdot 2 \cdot \sqrt{\left(\tfrac{3}{2}\right)^2 - y^2}\right)\left(0.5 - (-y)\right)\, dy$$

$$W = \rho g \int^{0} 12 \sqrt{\tfrac{9}{4} - y^2}\ (0.5 + y)\, dy$$

$$W = \rho g \int_{-1.5}^{0} 12\sqrt{\frac{9}{4}-y^2}\,(0.5+y)\,dy$$

15. A rope that is 20 feet long and weighs 2 pounds per foot supports a 160-lb weight while hanging over the side of a tall building. How much work, in ft-lb, would be required to pull the rope up 10 feet?

16. A spring has a natural length of 2 meters. If a force of 12 Newtons is required to hold the spring stretched to a length of 4 meters, find the work that would be required to stretch the spring from 3 to 7 meters.

17. Find the area between the curves $y = x^2 + 1$ and $y = x + 3$ from $x = 0$ to $x = 3$.

18. The region bounded by the curves $y = x - x^2$ anf the x-axis is rotated about the y-axis. Find the volume of the resultant solid.

πr^2 / $\pi(R^2 - r^2)$ $2\pi r h$

axis	Disks/washers	Shells

axis	Disks/washers	Shells
→	$\int dx$	$\int dy$
↑	$\int dy$	$\int dx$

$x=\sqrt{y}$

19. Which of the following integrals gives the volume of the solid formed by rotating the region bounded by the $y = x^2$ and $y = \sqrt[3]{x}$ about the line $y = -1$?

$x = y^3$

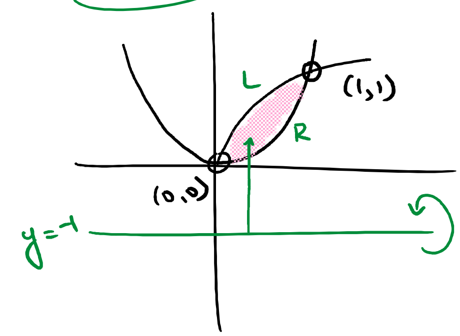
(a) $2\pi \int_0^1 (y-1)(\sqrt{y} - y^3)\, dy$

(b) $\pi \int_0^1 (y^3 - \sqrt{y})^2\, dy$

(c) $2\pi \int_0^1 (y+1)(\sqrt{y} - y^3)\, dy$

(d) $\pi \int_0^1 \left((x^2-1)^2 - (\sqrt[3]{x} - 1)^2\right) dx$

(e) $\pi \int_0^1 (x^2 - \sqrt[3]{x})(x+1)\, dx$



$h = R - L = \sqrt{y} - y^3$

20. Which of the following integrals gives the volume of the solid formed by rotating the region bounded by the $y = x$ and $y = x^2$ about the line $x = 2$?

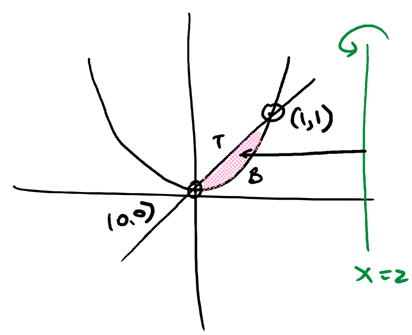
(a) $2\pi \int_0^1 (2-x)(x-x^2)\, dx$

(b) $2\pi \int_0^1 (2-y)(y-\sqrt{y})\, dy$

(c) $2\pi \int_0^1 (x-2)(x-x^2)\, dx$

(d) $\pi \int_0^1 (y-\sqrt{y})^2\, dy$

(e) $\pi \int_0^1 \left((2-x)^2 - (2-x^2)^2\right) dx$



2π

$r = 2 - x$

$h = x - x^2$