Exam 1 review (1.1 - 1.4, 2.1 - 2.2)

Problem 1. Given the function $f(x) = x^2 + 2x - 15$,

(1) What is the average rate of change of $f(x)$ on the interval $[5, 7]$?

$f(5)$ from $x=5$ to $x=7$ $f(7)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(7) - f(5)}{7-5}$$

$$= \frac{[7^2 + 2(7) - 15] - [5^2 + 2(5) - 15]}{2} = \frac{48 - 20}{2}$$

$$= \frac{28}{2} = 14$$

(2) What is the instantaneous rate of change of $f(x)$ at $x = 2$? $\rightarrow f'(2) = ?$

derivative $= f'(x=2)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$(a \pm b)^2 = (a+b)(a+b)$
$= a^2 \pm 2ab + b^2$

① $f(x) = x^2 + 2x - 15 \longrightarrow f(2)$

② $f(x+h) = (x+h)^2 + 2(x+h) - 15 \longrightarrow f(2+h)$

$$= x^2 + 2xh + h^2 + 2x + 2h - 15$$

③ $$f'(x) = \lim_{h\to 0} \frac{(\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h - \cancel{15}) - (\cancel{x^2} + \cancel{2x} - \cancel{15})}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x + h + 2)}{\cancel{h}} = \lim_{h\to 0} (2x + \cancel{h}^{0} + 2)$$

$$\boxed{f'(x) = 2x+2}\Bigg|_{x=2}$$

$f'(2) = 2(2) + 2 = 6$ Ans.

$$\boxed{(a-b)(a+b) = a^2 - b^2}$$

when you have a radical $\longrightarrow$ use conjugate

$\sqrt{3x+3h+1} - \sqrt{3x+1} \xrightarrow{\text{conjugate}} \sqrt{3x+3h+1} + \sqrt{3x+1}$

Problem 2. Given the function $f(x) = \sqrt{3x+1}$,

(1) Use the limit definition of the derivative to find $f'(x)$.

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

$f(x) = \sqrt{3x+1}$

$f(x+h) = \sqrt{3(x+h)+1} = \sqrt{3x+3h+1}$

$$= \lim_{h\to 0}\left(\frac{\sqrt{3x+3h+1} - \sqrt{3x+1}}{h}\right)\left(\frac{\sqrt{3x+3h+1}+\sqrt{3x+1}}{\sqrt{3x+3h+1}+\sqrt{3x+1}}\right)$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{3x+3h+1}\right)^2 - \left(\sqrt{3x+1}\right)^2}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)} = \frac{(\cancel{3x}+3h+\cancel{1}) - (\cancel{3x}+\cancel{1})}{h\left(\sqrt{3x+3h+1}+\sqrt{3x+1}\right)}$$

$$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}\left(\sqrt{3x+\cancel{3h}+1}+\sqrt{3x+1}\right)} = \frac{3}{\sqrt{3x+1}+\sqrt{3x+1}} = \boxed{\frac{3}{2\sqrt{3x+1}} = f'(x)}$$

$$\lim_{h\to 0} \frac{3}{\sqrt{3x+\cancel{3h}+1}+\sqrt{3x+1}}$$

m

★ (2) Find the equation of the tangent line to the graph of $f(x)$ at $x = 5$. $\longrightarrow$ $y\ (x=5)$

$f(x) = \sqrt{3x+1}$

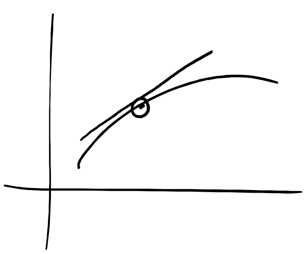
$f(5) = \sqrt{15+1}$

$= \sqrt{16}$

$= 4$

Eqn of a line $\longrightarrow$ pt & slope.

Pt: $(5, 4)$ $\quad (x_1, y_1)$



slope $= m = f'(x=5)$

$$f'(5) = \frac{3}{2\sqrt{3(5)+1}} = \frac{3}{2\sqrt{16}}$$

$$= \frac{3}{2\cdot 4} = \frac{3}{8} = m$$

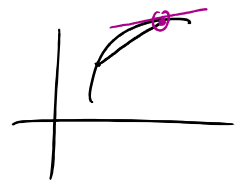
pt-slope form.

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \left(\frac{3}{8}\right)(x-5) = \frac{3x}{8} - \frac{15}{8}$$

$+4 \qquad +4$

$$y = \frac{3x}{8} - \frac{15}{8} + 4 = \frac{3x}{8} + \frac{17}{8}$$



Problem 3. List the different ways

(1) to describe the slope of a secant line

a) average rate of change : $\frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{f(b) - f(a)}{b - a}$.

b) difference quotient (DQ) : $\frac{f(x+h) - f(x)}{h}$

c) average velocity

(2) to describe the slope of a tangent line

a) instantaneous rate of change / inst. velocity

b) rate of change

c) limit of DQ $\longrightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

d) limit of slopes of secant line

e) derivative : $f'(x)$

f) slope m

(3) that a function can be non-differentiable.

can't take derivative here

dis-Continuity

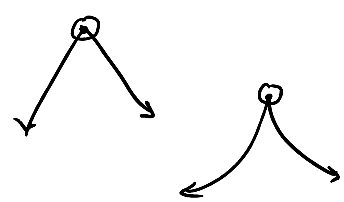
LHL = RHL = $f(x)$

a) holes

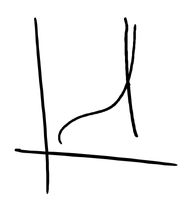
b) VA

c) jump

corner or cusp

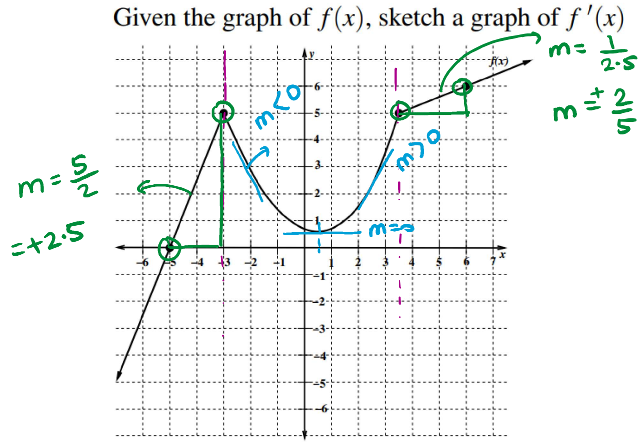


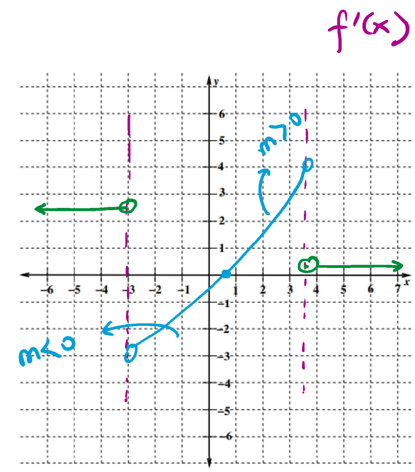
vertical tangent line



Problem 4. .

Given the graph of $f(x)$, sketch a graph of $f'(x)$





Problem 5. Find all the vertical asymptotes and holes for $f(x) = \dfrac{(x+2)(x-3)(x-8)^2}{(x-8)^3(x-5)(x-3)}$

$$f(x) = \frac{(x+2)\cancel{(x-3)}\cancel{(x-8)}\cancel{(x-8)}}{\cancel{(x-8)}\cancel{(x-8)}(x-8)(x-5)\cancel{(x-3)}}$$

$$= \frac{(x+2)}{(x-8)(x-5)}$$

$x=8$

$\underbrace{8^- - 8}_{}$

$7.999 - 8$

$= -0.001$

$\underbrace{8^+ - 8}_{}$

$8.001 - 8$

$= +0.001$

Vertical Asymptotes : $\longrightarrow$ x values that cause a division by zero

$x-8=0 \longrightarrow x=8$

$x-5=0 \longrightarrow x=5$

Holes: $x-3=0 \longrightarrow x=3$

$x=8$

LHL $\displaystyle\lim_{x\to 8^-} \frac{(x+2)}{(x-8)(x-5)} = \frac{10}{(0^-)(3)} \to -\infty$

RHL $\displaystyle\lim_{x\to 8^+} \frac{(x+2)}{(x-8)(x-5)} = \frac{10}{(0^+)(3)} \to +\infty$

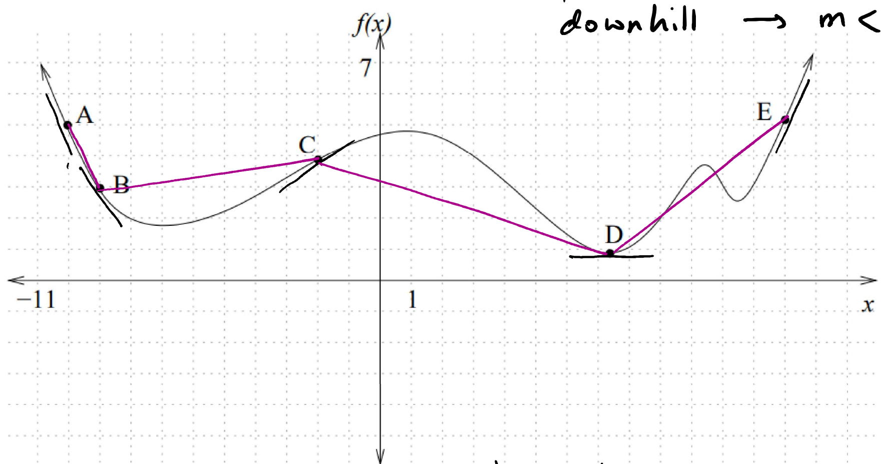
$x=8$

$-0.00001 \leftarrow \underset{x=0}{|} \rightarrow +0.00001$

$\frac{1}{0.00001} = 10000$

Problem 6. Given the graph of $f(x)$ below

walking from left to right
uphill $\rightarrow m > 0$
downhill $\rightarrow m < 0$



slope of tangent line to the curve.

(1) At which labeled point(s) is the derivative positive?

C, E

(2) At which labeled point(s) is the derivative negative?

A, B

(3) At which labeled point(s) is the derivative zero?

D

(4) At which labeled point(s) is the derivative largest?

E

(5) At which labeled point(s) is the derivative smallest?

A

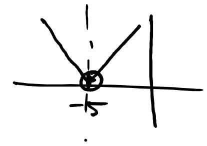
(6) Between which two labeled points is the average rate of change largest?

D & E

$a^2 - b^2 = (a+b)(a-b)$

Problem 7. Find the following limits if they exist

(1) $\displaystyle\lim_{x\to -5^-} \frac{|x+5|}{x^2-25}$



$g(x) = |x+5|$

$x+5=0$

$x=-5$

$x \to (-5^-)$

-5^-

$$|x+5| = \begin{cases} -(x+5), & x<-5 \\ (x+5), & x \geq -5 \end{cases}$$

$$= \lim_{x\to -5^-} \frac{-(x+5)}{x^2-25} \sim \frac{0}{0}$$

$$= \lim_{x\to -5^-} \frac{-\cancel{(x+5)}}{\cancel{(x+5)}(x-5)} = \lim_{x\to -5^-} \frac{-1}{(x-5)}$$

$$= \frac{-1}{(-5-5)} = \frac{-1}{-10} = \frac{1}{10}$$

Ans.

$\frac{e^x}{e^{-x}} = e^x \cdot e^x = e^{2x}$

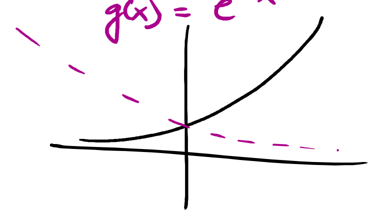
$$\longrightarrow \lim_{x\to -5^+} \frac{|x+5|}{x^2-25} = -\frac{1}{10}$$

(2) $\displaystyle\lim_{x\to -\infty} \frac{2e^x - 11}{3e^{-x} + 5e^x + 2}$

largest term: e^{-x}

$f(x) = e^x$

$g(x) = e^{-x}$



$$\lim_{x\to(-\infty)} \frac{\frac{2e^x}{e^{-x}} - \frac{11}{e^{-x}}}{3\frac{\cancel{e^{-x}}}{\cancel{e^{-x}}} + \frac{5e^x}{e^{-x}} + \frac{2}{e^{-x}}}$$

$\displaystyle\lim_{x\to\infty} e^x \longrightarrow \infty$ $\quad \displaystyle\lim_{x\to\infty} e^{-x} = 0$

$\displaystyle\lim_{x\to-\infty} e^x = 0$ $\quad \displaystyle\lim_{x\to(-\infty)} e^{-x} \to \infty$

$$= \lim_{x\to(-\infty)} \frac{2e^{2x} - 11e^x}{3 + 5e^{2x} + 2e^x}$$

$$\frac{0-0}{3+0+0} = \frac{0}{3} = 0$$

Problem 8. The monthly revenue of a local candy shop is given by $R(x) = -x^2 + \cancel{12x}$ $120x$ dollars when x gift baskets are sold each month. The shop's monthly cost function, $C(x) = 40x + 1500$ dollars when x gift baskets are made each month.

(1) Find the average rate of change of revenue when the number of gift baskets sold each month changes from 35 to 40 baskets. Interpret your answer.

$$m = \frac{R(40) - R(35)}{40-35} = \frac{3200 - 2975}{5} = \frac{225}{5} = 45$$

Ans: When the number of gift baskets sold increases from 35 to 40 baskets, the revenue increases on average by \$45/basket.

(2) Find the rate of change of profit when 35 gift baskets are made and sold each month. Interpret your answer.

$$P(x) = R(x) - C(x) = (-x^2 + 120x) - (40x + 1500)$$
$$= -x^2 + 120x - 40x - 1500$$
$$P(x) = -x^2 + 80x - 1500$$

rate of change @ 35

$$P'(35) = \lim_{h\to 0} \frac{P(35+h) - P(35)}{h}$$

$$P(35) = -(35)^2 + 80(35) - 1500$$
$$P(35+h) = -(35+h)^2 + 80(35+h) - 1500$$
$$= -(35^2 + 2(35)h + h^2) + 80(35) + 80h - 1500$$

$$P'(35) = \lim_{h\to 0} \frac{\left(-(\cancel{35^2} + 2(35)h + h^2) + \cancel{80(35)} + 80h - \cancel{1500}\right) - \left(-\cancel{35^2} + \cancel{80(35)} - \cancel{1500}\right)}{h}$$

$$= \frac{-2(35)h + h^2 + 80h}{h} = \lim_{h\to 0} \frac{\cancel{h}\left[-70 + h + 80\right]}{\cancel{h}}$$

$$= \lim_{h\to 0} (-70 + \cancel{h}^{0} + 80) = -70 + 80 = 10$$

$$= \lim_{h\to 0} (-70 + \cancel{h}^{0} + 80) = -70 + 80 = 10$$

Ans: when 35 baskets are made and sold, the profit increases by \$10/basket.

Problem 9. Given the function $f(x) = \dfrac{5x}{3x-4}$

(1) find the equation of the secant line from the point $\underbrace{(-2, 1)}_{(x_1, y_1)}$ to the point $\underset{(x_2,y_2)}{(1, -5)}$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-1}{1-(-2)} = \frac{-6}{1+2} = \frac{-6}{3} = -2$$

use pt $(-2, 1)$

$$\boxed{y = -2x - 3}$$

Eqn: $y - y_1 = m(x - x_1)$

$$y - 1 = (-2)(x - -2) = (-2)(x+2) = -2x - 4$$

$+1 \qquad +1$

(2) find the equation of the tangent line at the point $\boxed{(1, -5)}$.

$x = 1$, $y = -5$

$$m = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = \frac{5x}{3x-4} \qquad f(x+h) = \frac{5(x+h)}{3(x+h)-4} = \frac{5x+5h}{3x+3h-4}$$

$$m = \lim_{h\to 0} \frac{\frac{5x+5h}{3x+3h-4} - \frac{5x}{3x-4}}{h} = \frac{\frac{(5x+5h)(3x-4)}{(3x+3h-4)(3x-4)} - \frac{(5x)(3x+3h-4)}{(3x+3h-4)(3x-4)}}{h}$$

$$= \frac{1}{h}\left[\frac{(\cancel{15x^2} - \cancel{20x} + \cancel{15xh} - 20h) - (\cancel{15x^2} + \cancel{15xh} - \cancel{20x})}{(3x+3h-4)(3x-4)}\right]$$

$$m\Big|_{x=1} = \lim_{h\to 0} \frac{-20\cancel{h}}{\cancel{h}(3x+3h-4)(3x-4)} = \frac{-20}{(3x-4)(3x-4)}\Big|_{x=1}$$

$$= \frac{-20}{(3(1)-4)(3(1)-4)} = \frac{-20}{+1}$$

$m = -20$ $\qquad$ pt $= (1, -5)$

$$y - y_1 = m(x - x_1) \Rightarrow y - -5 = (-20)(x-1)$$

$y + 5 = -20x + 20$
$\quad -5 \qquad\qquad -5$

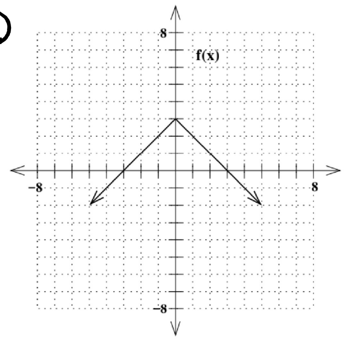
$$\boxed{y = -20x + 15}$$

9

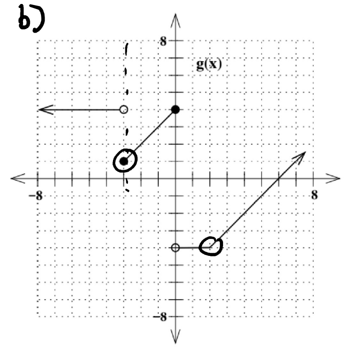
Non D.

Problem 10. Where are the following functions continuous? Where are they non-differentiable?

a)

(1) 

b)



D: $(-\infty, \infty)$

Non D @ $x = 0$

D: $(-\infty, -3) \cup (-3, 0) \cup (0, \infty)$

Non D: @ $x = -3, 0$ and $x = 2$

(2) $f(x) = \begin{cases} \dfrac{x+1}{2x^2 - 3x - 9} & \text{if } x < 1 \quad 1^- \\ x^2 + 1 & \text{if } x \geq 1 \quad 1^+ \end{cases}$

$$\begin{cases} f(1^-) = \dfrac{1^- + 1}{2(1^-)^2 - 3(1^-) - 9} = \dfrac{2}{2 - 3 - 9} = \dfrac{2}{-10} \\ f(1^+) = (1^+)^2 + 1 = 2 \end{cases}$$

$-3/2 \qquad x = 1 \qquad \cancel{3}$

Non D @ $x = 1 \rightarrow$ jump

@ $x = -3/2 \rightarrow$ VA

D: $(-\infty, -3/2) \cup (-3/2, 1) \cup (1, \infty)$

(3) $f(x) = \dfrac{x-5}{\sqrt{x+1}} + \ln(3x-7)$

① & ② $\rightarrow \quad x + 1 > 0$

$x > -1$

$-1 \qquad 0 \qquad 7/3$

$\ln(3x - 7)$

③ $\quad 3x - 7 > 0$

$3x > 7$

$x > 7/3$

D: $\left(\frac{7}{3}, \infty\right)$

non D @ $x \leq 7/3$

$2x^2 - 3x - 9$

$x^2 - 3x - (9)(2)$

$x^2 - 3x - 18$

$(x - \frac{6}{2})(x + \frac{3}{2})$

$\underbrace{(x-3)}\underbrace{(x + 3/2)}$

$x - 3 = 0$

$\cancel{x = 3}$

outside D of first piece

$x + 3/2 = 0$

$x = -3/2 \rightarrow$ VA

$f(x)$ for $x < 1$

$$f(x) = \frac{x+1}{2x^2 - 3x - 9}$$

$$= \frac{x+1}{(x-3)(x+3/2)} \quad \text{for } x < 1$$

outside D.

VA