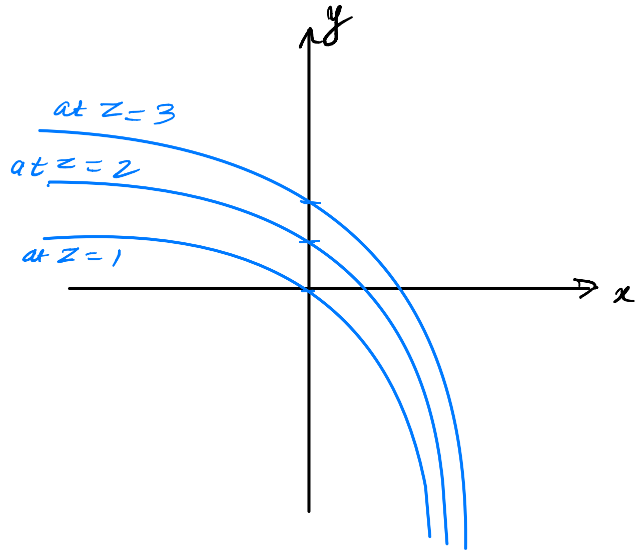
Note: As sections 14.1 - 14.5 were covered in the WIR session last week, this WIR session focuses on the remaining sections (that is, 14.6 - 14.8). Students are strongly encouraged to review last week's WIR session.

Example 1 (14.1)**.** *Sketch the level curves of*



(a) $f(x, y) = e^x + y$ *at* $z = 1, 2, 3.$

Let $z = c$, a constant in the range of f.

Then $e^x + y = c \Rightarrow \boxed{y = -e^x + c}$

In particular, $c = 1 \Rightarrow y = -e^x + 1$

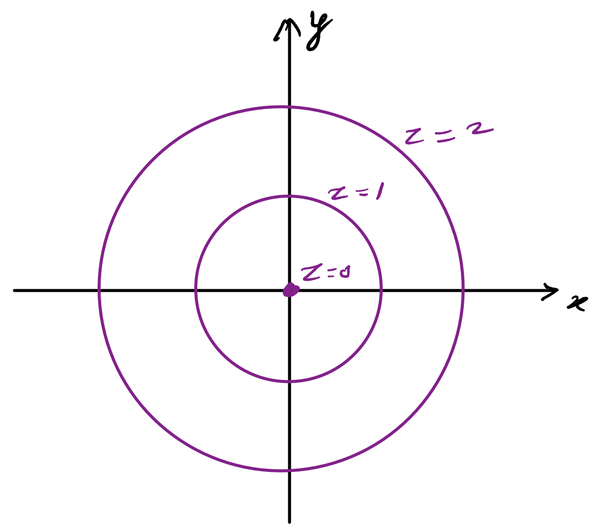
$c = 2 \Rightarrow y = -e^x + 2$

$c = 3 \Rightarrow y = -e^x + 3$

(b) $f(x, y) = e^{x^2+y^2}$ *at* $z = 1, 2, 3.$

$z = c$ with $c \geq 1 \Rightarrow c = e^{x^2+y^2}$

$\Rightarrow \boxed{x^2 + y^2 = \ln c}$



In particular, $c = 1 \Rightarrow x^2 + y^2 = 0$

$c = 2 \Rightarrow x^2 + y^2 = \ln 2$

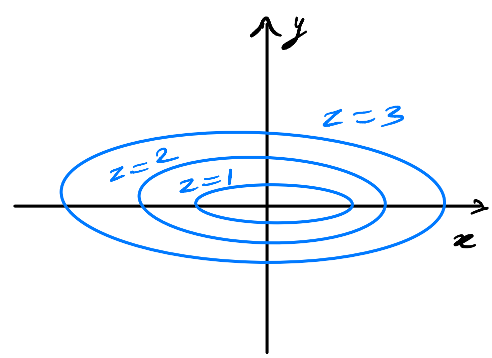
$c = 3 \Rightarrow x^2 + y^2 = \ln 3$

(b) $f(x, y) = \ln(x^2 + 9y^2)$ *at* $z = 1, 2, 3.$

$z = c$, where c is a constant implies

$\ln(x^2 + 9y^2) = c$

$\Rightarrow x^2 + 9y^2 = e^c \Rightarrow \boxed{\dfrac{x^2}{e^c} + \dfrac{y^2}{e^c/9} = 1}$



In particular, $z = c = 1 \Rightarrow \dfrac{x^2}{e} + \dfrac{y^2}{e/9} = 1$

$z = c = 2 \Rightarrow \dfrac{x^2}{e^2} + \dfrac{y^2}{e^2/9} = 1$

$\vdots$

Example 2 (14.4). *Consider the function*

$$f(x,y) = ye^{xy}. \longrightarrow f(0,3) = 3$$

(a) Find the linearization of the function at the point $(0,3)$. ($a = 0$, $b = 3$)

(b) Use differentials or the linearization to estimate $(2.98)e^{(0.03)(2.98)}$.

(a) $f_x(x,y) = y^2 e^{xy} \Rightarrow f_x(0,3) = 9$

$f_y(x,y) = e^{xy} + xye^{xy} \Rightarrow f_y(0,3) = 1$

The linearization f at (a,b) is

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= 3 + 9(x-0) + 1(y-3)$

$$\boxed{L(x,y) = 9x + y}$$

(b) $(2.98)e^{(0.03)(2.98)} = f(0.03, 2.98) \approx L(0.03, 2.98)$

$= 9(0.03) + 2.98$

$= 3.25$

Example 3 (14.4). *The radius and height of a right circular cone are measured as 6 ft and 10 ft, respectively, with a possible error of at most 0.1 ft. Use differentials to estimate the maximum error in the calculated volume of the cone.*

$V = \frac{1}{3}\pi r^2 h$. $\quad dV = ?$ when $r=6, h=10, dr = dh = 0.1$.

$$dV = V_r \cdot dr + V_h \cdot dh$$
$$= \frac{2\pi r h}{3} \cdot dr + \frac{\pi r^2}{3} \cdot dh$$
$$= \frac{2\pi}{3} \cdot (6)(10)(0.1) + \frac{\pi}{3} \cdot (36)(0.1)$$
$$= 4\pi + 1.2\pi$$
$$= 5.2\pi \ ft^3.$$

Example 4 (14.5). *Let* $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $x = re^s$, $y = se^r$ *and* $z = e^{rs}$. *Find* $\frac{\partial f}{\partial r}$ *when* $r = 0$ *and* $s = 2$. $\Rightarrow x = 0, y = 2, z = 1$.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$$
$$= \frac{x}{\sqrt{x^2+y^2+z^2}} \cdot e^s + \frac{y}{\sqrt{x^2+y^2+z^2}} \cdot se^r + \frac{z}{\sqrt{x^2+y^2+z^2}} \cdot se^{rs}$$
$$= 0 + \frac{2}{\sqrt{5}}(2e^0) + \frac{1}{\sqrt{5}}\cdot(2e^0) = \frac{6}{\sqrt{5}}.$$

Try $\left.\frac{\partial f}{\partial s}\right|_{r=0 \text{ and } s=2}$.

Example 5 (14.5). *Find* $\dfrac{\partial z}{\partial x}$ *and* $\dfrac{\partial z}{\partial y}$ *if* $x^2 + y^2 + z^2 = 2e^{xyz}$.

Define $F(x,y,z) = x^2+y^2+z^2-2e^{xyz}$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{(2x - 2yz\,e^{xyz})}{(2z - 2xy\,e^{xyz})}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{(2y - 2xz\,e^{xyz})}{(2z - 2xy\,e^{xyz})}$$

Example 6 (14.6). *Find the directional derivative of* $f(x,y) = x\sin(xy)$ *at* $P(1,\pi)$ *in the direction of the vector* $\mathbf{v}$ *that makes an angle* $\theta = \pi/3$ *with positive* x*-axis.*

$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle \sin xy + xy\cos(xy),\ x^2\cos(xy) \rangle$

$\nabla f(1,\pi) = \langle -\pi, -1 \rangle$

The unit vector in the direction of v is $u = \cos\theta\, i + \sin\theta\, j$

i.e. $u = \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$$D_u f(1,\pi) = \nabla f(1,\pi)\cdot u$$
$$= \langle -\pi, -1 \rangle \cdot \langle \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle$$
$$= \frac{-(\pi+\sqrt{3})}{2}$$

Example 7 (14.1/14.6). *Suppose* $f(x, y) = 2xy + \ln(4x + y)$.

(a) Sketch the domain of the function.

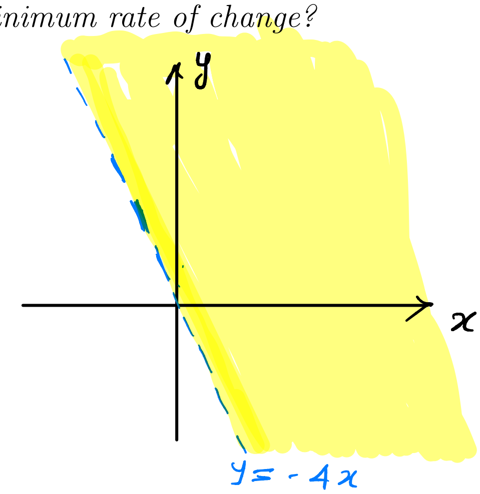
(b) Find the directional derivative of f *at* $P(-1/4, 2)$ *in the direction from* P *to* $Q(3/4, 1)$.

(c) In what direction does f *increase fastest at* P? *What is the maximum rate of change?*

(d) In what direction does f *decrease fastest at* P? *What is the minimum rate of change?*

(a) Only the restriction is $4x + y > 0$.

So, $\boxed{y > -4x}$



(b) $\overrightarrow{PQ} = \langle \frac{3}{4} + \frac{1}{4}, 1 - 2 \rangle = \langle 1, -1 \rangle$

$\vec{v} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \dfrac{1}{\sqrt{2}} \langle 1, -1 \rangle$

$$\nabla f(x, y) = \left\langle 2y + \frac{4}{4x + y}, \ 2x + \frac{1}{4x + y} \right\rangle$$

$$\nabla f(-\tfrac{1}{4}, 2) = \left\langle 4 + \frac{4}{1}, \ 2 \cdot (-\tfrac{1}{4}) + \frac{1}{1} \right\rangle = \left\langle 8, \frac{1}{2} \right\rangle$$

$$D_v f(-\tfrac{1}{4}, 2) = \left\langle 8, \frac{1}{2} \right\rangle \cdot \frac{1}{\sqrt{2}} \langle 1, -1 \rangle = \frac{1}{\sqrt{2}} \left[8 - \frac{1}{2} \right] = \frac{15}{2\sqrt{2}}$$

(c) The function f increases fastest at P in the direction of $\langle 8, \frac{1}{2} \rangle$ (or $\langle 16, 1 \rangle$), and the maximum rate of change is $|\langle 8, \frac{1}{2} \rangle| = \sqrt{64 + \frac{1}{4}} = \dfrac{\sqrt{257}}{2}$.

(d) The function f decreases fastest at P in the direction of $-\langle 8, \frac{1}{2} \rangle = \langle -8, -\frac{1}{2} \rangle$ (or $\langle -16, -1 \rangle$), and the minimum rate of change is $-|\langle 8, \frac{1}{2} \rangle| = -\dfrac{\sqrt{257}}{2}$.

Example 8 (14.6). *Find equations of (a) the tangent plane and (b) the normal line to the surface $x^2 + y^2 + yz = xz^2$ at the point $P(1, -1, 1)$.*

Define $F(x,y,z) = x^2+y^2+yz-xz^2$. Then the level surface $F(x,y,z)=0$ produces the given surface.

$\nabla F(x,y,z) = \langle 2x - z^2, 2y+z, y-2xz \rangle$.

$\nabla F(1,-1,1) = \langle 1, -1, -3 \rangle$

(a) A normal vector for the tangent plane to the given surface at $P(1,-1,1)$ is $\vec{n} = \nabla F(1,-1,1) = \langle 1,-1,-3 \rangle$.

So, an equation of the tangent plane is

$$\langle 1,-1,-3 \rangle \cdot \langle x-1, y-(-1), z-1 \rangle = 0$$

$$x-1-y-1-3z+3=0$$

$$\boxed{x-y-3z=-1}$$

(b) A direction vector for the normal line at $P(\overset{x_0}{1},\overset{y_0}{-1},\overset{z_0}{1})$ is

$\vec{v} = \nabla F(1,-1,1) = \langle \overset{a}{1}, \overset{b}{-1}, \overset{c}{-3} \rangle$.

So, parametric equations for the normal line are

$$x = x_0 + at = 1+t$$

$$y = y_0 + bt = -1-t$$

$$z = z_0 + ct = 1-3t$$

Example 9 (14.7). *Find the local minimum and maximum values and saddle points of the function*

$$f(x, y) = 3xy - x^2y - xy^2 + 2.$$

$f_x = 3y - 2xy - y^2$

$f_y = 3x - x^2 - 2xy$

$f_{xx} = -2y \quad f_{yy} = -2x$

$f_{xy} = 3 - 2x - 2y$

$D = f_{xx}f_{yy} - (f_{xy})^2 = 4xy - (3-2x-2y)^2$

Critical points:

① $\leftarrow f_x = 0 \Rightarrow y(3-2x-y) = 0 \Rightarrow y = 0$ or $y = 3-2x$

② $\leftarrow f_y = 0 \Rightarrow 3x - x^2 - 2xy = 0 \Rightarrow x(3-x-2y) = 0$

subs. $y = 0$ into eqⁿ ② $\Rightarrow x(3-x) = 0 \Rightarrow x = 0$ or $x = 3$.

$\Rightarrow \boxed{(0,0), (3,0)}$

subs. $y = 3-2x$ into eqⁿ ② $\Rightarrow x(3-x-2(3-2x)) = 0$

$\Rightarrow x(3-x-6+4x) = 0$

$\Rightarrow x(3x-3) = 0 \Rightarrow x = 0$ or $x = 1$

And $x = 0 \Rightarrow y = 3 - 2\cdot 0 \rightsquigarrow (0,3)$

$x = 1 \Rightarrow y = 1 \rightsquigarrow (1,1)$

So, $\boxed{(0,0), (3,0), (0,3), (1,1)}$ are the critical points of f.

$D(0,0) = D(3,0) = D(0,3) = -9 < 0$.

So, $(0,0)$, $(3,0)$ and $(0,3)$ are saddle points.

$D(1,1) = 4 - (3-2-2)^2 = 3 > 0$,

$f_{xx}(1,1) = -2 < 0$. So, f has local max at $(1,1)$, which is $f(1,1) = 3 - 1 - 1 + 2 = 3$

Example 10 (14.7). *Find the absolute maximum and minimum values of $f(x,y) = x^2 + y^2 - 2x$ over a triangular region D with vertices $(0,2), (0,-2)$ and $(4,-2)$.*

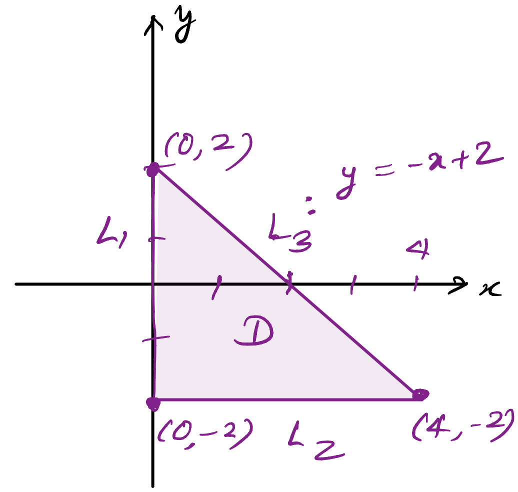
Critical point in D:

$0 = f_x = 2x - 2 \Rightarrow x = 1$

$0 = f_y = 2y = 0 \Rightarrow y = 0$

$(1,0)$ is the only critical point, which lies inside D.

$\boxed{f(1,0) = 1+0-2 = -1}$



Evaluate f along L_1: $x = 0 \Rightarrow f(0,y) = y^2$, $-2 \le y \le 2$.

$g(y) = y^2$ has minimum at $y=0$ and max at $y = \pm 2$.

So, $\boxed{\begin{array}{l} f(0,\pm 2) = 4 \\ f(0,0) = 0 \end{array}}$

Evaluate f along L_2: $y = -2 \Rightarrow f(x,-2) = x^2 - 2x + 4$, $0 \le x \le 4$

$g(x) = x^2 - 2x + 4 \Rightarrow g'(x) = 2x - 2 = 0$

$\Rightarrow x = 1$ is a critical number.

$\boxed{\begin{array}{l} f(1,-2) = 3 \\ f(4,-2) = 16 - 8 + 4 = 12 \end{array}}$

$2 \cdot \frac{9}{4_2} - \cancel{6}^{3} \cdot \frac{3}{\cancel{2}} + 4$

$4.5 + 4 - 9 = -6.5$

f along L_3: $y = -x+2 \Rightarrow f(x,-x+2) = x^2 + (-x+2)^2 - 2x = 2x^2 - 6x + 4$

$g(x) = 2x^2 - 6x + 4$, $0 \le x \le 4$.

$g'(x) = 0 \Rightarrow 4x - 6 = 0 \Rightarrow x = 3/2$. Need $g(0), g(4), g(3/2)$.

$g(3/2) = f(3/2, 1/2) = -1/2$

$g(0) = f(0,2)$
$g(4) = f(4,-2)$ } already computed.

$f(4,-2) = 12$ Max.

$f(1,0) = -1$ Min.

Example 11 (14.7). *Find the point on the plane*

$$x - 2y + 3z = 6$$

that is closest to the point $(0, 1, 1)$.

Let (x, y, z) be the point on the plane $x - 2y + 3z = 6$.

Then the distance from (x, y, z) to $(0, 1, 1)$ is

$$d^2 = (x-0)^2 + (y-1)^2 + (z-1)^2 \quad \text{——} \ (1)$$

But $x - 2y + 3z = 6 \Rightarrow z = \dfrac{6 - x + 2y}{3} = 2 - \dfrac{1}{3}x + \dfrac{2}{3}y$

substituting into (1),

$$d^2 = x^2 + (y-1)^2 + \left(1 - \frac{1}{3}x + \frac{2}{3}y\right)^2 \overset{\text{say}}{=} f(x, y).$$

We want to minimize f.

$$f_x = 2x + 2\left(1 - \frac{x}{3} + \frac{2y}{3}\right)\left(-\frac{1}{3}\right) = \frac{20}{9}x - \frac{4}{9}y - \frac{2}{3}$$

$$f_y = 2(y-1) + 2\left(1 - \frac{x}{3} + \frac{2y}{3}\right)\left(\frac{2}{3}\right) = -\frac{4}{9}x + \frac{26}{9}y - \frac{2}{3}$$

Solving $\left.\begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix}\right\}$ gives $(x, y) = \left(\frac{5}{14}, \frac{2}{7}\right)$, only one critical point.

$$z = 2 - \frac{1}{3}\left(\frac{5}{14}\right) + \frac{2}{3}\left(\frac{2}{7}\right) = \frac{29}{14}.$$

Thus, the closest point to $(0, 1, 1)$ on the plane is $\left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right)$.

Example 12 (14.8). *Use Lagrange multipliers method to find the point on the plane*

$$x - 2y + 3z = 6$$

that is closest to the point $(0, 1, 1)$.

Let (x, y, z) be the point on the plane $x - 2y + 3z = 6$.

Then the distance from (x,y,z) to $(0,1,1)$ is

$$d^2 = (x-0)^2 + (y-1)^2 + (z-1)^2 \overset{\text{say}}{=} f(x,y,z).$$

We want to minimize $f(x,y,z) = x^2 + (y-1)^2 + (z-1)^2$

subject to

$$g(x,y,z) = x - 2y + 3z = 6.$$

$$\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2(y-1), 2(z-1) \rangle = \lambda \langle 1, -2, 3 \rangle$$

① $\rightarrow$ $2x = \lambda$

② $\rightarrow$ $2(y-1) = -2\lambda \xRightarrow{\lambda = 2x} \cancel{2}(y-1) = -\overset{2}{\cancel{4}}x$ $\quad y = 1 - 2x$

③ $\rightarrow$ $2(z-1) = 3\lambda \xRightarrow{\lambda = 2x} 2(z-1) = 6x \Rightarrow z = 1 + 3x$

④ $\rightarrow$ $x - 2y + 3z = 6$

subs. $y = 1 - 2x$ and $z = 1 + 3x$ into ④;

$x - 2(1-2x) + 3(1+3x) = 6$

$x - 2 + 4x + 3 + 9x = 6 \Rightarrow 14x = 5 \Rightarrow \boxed{x = \frac{5}{14}}$

and so, $y = 1 - 2\left(\frac{5}{14}\right) = \frac{4}{14} = \frac{2}{7} \rightsquigarrow \boxed{y = \frac{2}{7}}$

$z = 1 + 3\left(\frac{5}{14}\right) = \frac{29}{14} \rightsquigarrow \boxed{z = \frac{29}{14}}$

f is minimized at $\left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right)$. That is, the point on the plane closest to $(0,1,1)$ is $\left(\frac{5}{14}, \frac{2}{7}, \frac{29}{14}\right)$.

Example 13 (14.8). *Use Lagrange multipliers to find the extreme values the function*

$$f(x, y) = 2xe^y + 5$$

subject to $x^2 + y^2 = 2$.

$g(x,y) = x^2+y^2$

$\nabla f = \lambda \nabla g \Rightarrow \langle 2e^y, 2xe^y \rangle = \lambda \langle 2x, 2y \rangle$

① $\rightarrow$ $\cancel{2}e^y = \cancel{2}\lambda x$ As $e^y > 0$, $x \neq 0$, $\lambda \neq 0$.

② $\rightarrow$ $\cancel{2}xe^y = \cancel{2}\lambda y$ so, $\lambda = \frac{e^y}{x}$.

③ $\rightarrow$ $x^2 + y^2 = 2$

Substituting $\lambda = \frac{e^y}{x}$ into ②, $x\cancel{e^y} = \frac{\cancel{e^y}}{x} \cdot y$

$\Rightarrow \boxed{y = x^2}$

Substituting $y = x^2$ into ③ gives

$x^2 + x^4 = 2$

$x^4 + x^2 - 2 = 0$

$(x^2+2)(x^2-1) = 0$

$\Rightarrow x^2 - 1 = 0 \Rightarrow \boxed{x = \pm 1.}$

$y = x^2 \Rightarrow \boxed{y = 1}$

So, f has extreme values at $(1, 1)$ and $(-1, 1)$.

$f(1,1) = 2e + 5 \leftarrow$ Max

$f(-1,1) = -2e + 5 \leftarrow$ Min