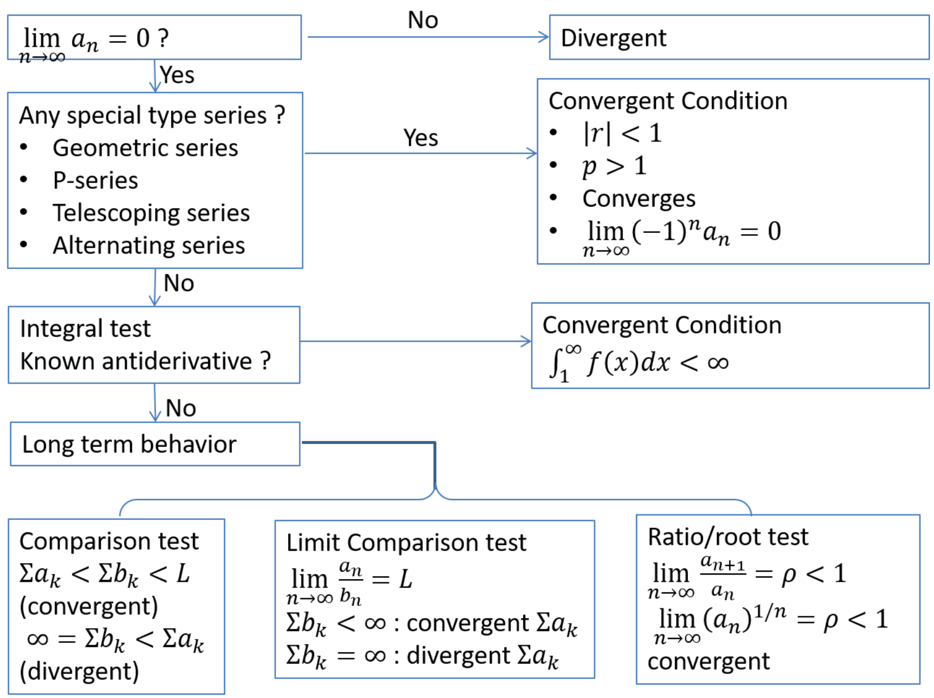
Week in Review
Math 152

Week 10

Alternating series

Absolute Convergence and the Ratio Test

Divergence test



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

Explain why following series do NOT converge

1. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

2. $\sum_{n=1}^{\infty} (-1)^n$

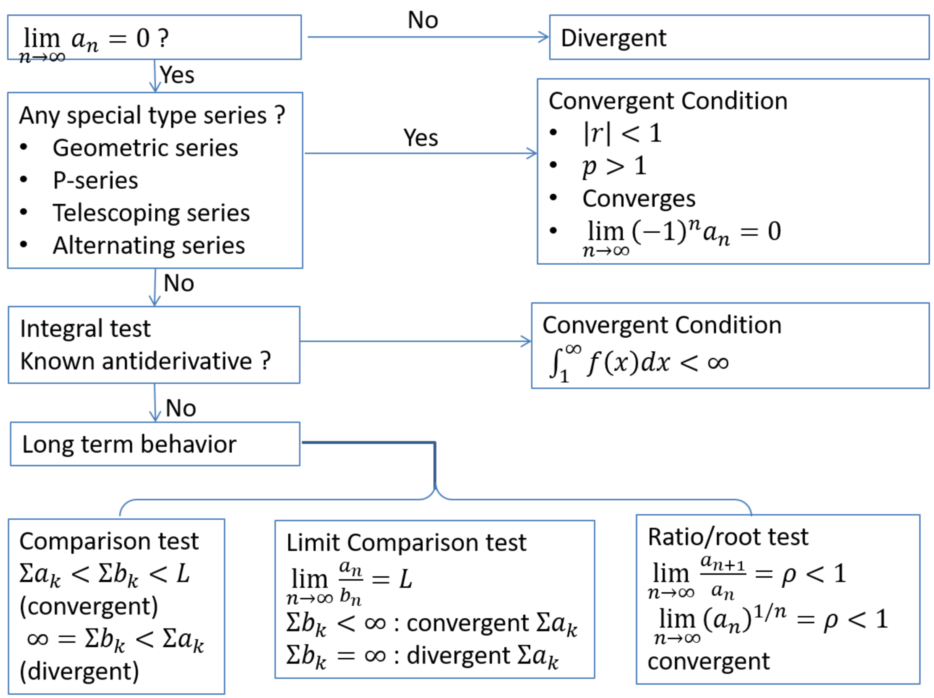
3. $\sum_{n=1}^{\infty} \frac{1}{e^{1/n}}$

By divergence test

- $\lim_{n\to\infty} \frac{n}{n+1} = 1$
- $\lim_{n\to\infty} (-1)^n = DNE$
- $\lim_{n\to\infty} \frac{1}{e^{1/n}} = 1$

Week 10

Geometric series



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

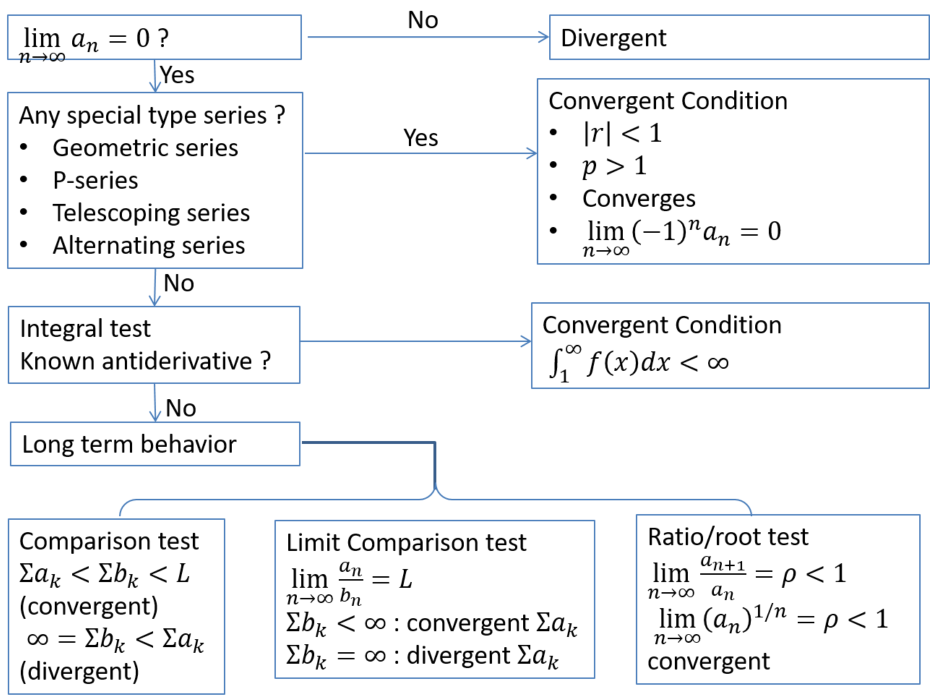
Evaluate $\sum_{n=1}^{\infty} \frac{1}{e^n}$

Geometric series with
$a = \frac{1}{e}$ and $r = \frac{1}{e}$
Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e-1}$$

Week 10

Geometric series



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

Evaluate

1. $\sum_{n=1}^{\infty} \frac{1}{e^n}$

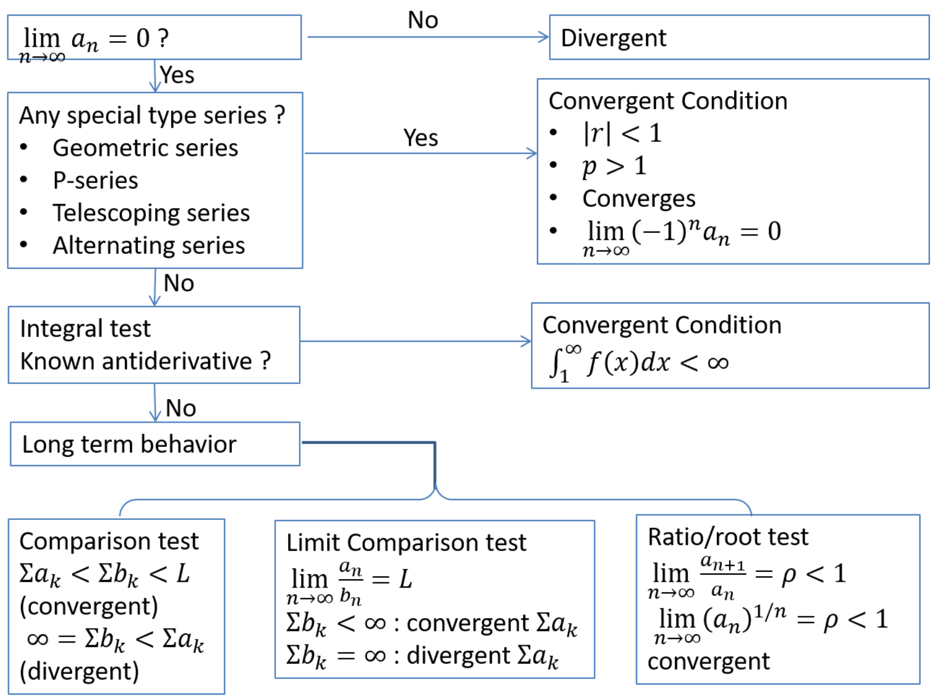
2. $\sum_{n=1}^{\infty} \frac{5}{2^{n-1}}$

1. Geometric series with
$a = \frac{1}{e}$ and $r = \frac{1}{e}$
Therefore,
$$\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{1}{e-1}$$

2. Geometric series with
$a = 5$ and $r = \frac{1}{2}$
Therefore,
$$\sum_{n=1}^{\infty} \frac{5}{2^{n-1}} = \frac{5}{1-\frac{1}{2}} = 10$$

Week 10

P- series



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

Evaluate

1. $\sum_{n=1}^{\infty} \frac{1}{n}$

2. $\sum_{n=1}^{\infty} \frac{5}{n^{1.1}}$

P-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

1. $p = 1 \leq 1 \Rightarrow$ Divergent

2. $p = 1.1 > 1 \Rightarrow$ Convergent

Week 10

Telescoping series

$\lim_{n\to\infty} a_n = 0$?

- No → Divergent
- Yes ↓

Any special type series ?

- Geometric series
- P-series
- Telescoping series
- Alternating series

Yes → Convergent Condition

- $|r| < 1$
- $p > 1$
- Converges
- $\lim_{n\to\infty} (-1)^n a_n = 0$

No ↓

Integral test
Known antiderivative ?

→ Convergent Condition
$\int_1^\infty f(x)dx < \infty$

No ↓

Long term behavior

Comparison test
$\Sigma a_k < \Sigma b_k < L$
(convergent)
$\infty = \Sigma b_k < \Sigma a_k$
(divergent)

Limit Comparison test
$\lim_{n\to\infty} \frac{a_n}{b_n} = L$
$\Sigma b_k < \infty$: convergent Σa_k
$\Sigma b_k = \infty$: divergent Σa_k

Ratio/root test
$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$
$\lim_{n\to\infty} (a_n)^{1/n} = \rho < 1$
convergent

+/- terms but Not alternating
⇒ Absolute convergence

Evaluate

1. $\sum_{n=1}^{\infty} \left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right]$

2. $\sum_{n=1}^{\infty} \left[\ln\left(\frac{1}{n}\right) - \ln\left(\frac{1}{n+1}\right)\right]$

Telescoping series

1. $\sum_{n=1}^{\infty} \left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right]$

$$= \lim_{n\to\infty} \left(\cos 1 - \cos\frac{1}{n+1}\right) = \cos 1 - 1$$

2. $\sum_{n=1}^{\infty} \left[\ln\left(\frac{1}{n}\right) - \ln\left(\frac{1}{n+1}\right)\right]$

$$= \lim_{n\to\infty} \left(\ln 1 - \ln\frac{1}{n+1}\right) = \infty$$

Week 10

Integral test

$\lim_{n\to\infty} a_n = 0$?

- No → Divergent
- Yes ↓

Any special type series ?
- Geometric series
- P-series
- Telescoping series
- Alternating series

Yes → Convergent Condition
- $|r| < 1$
- $p > 1$
- Converges
- $\lim_{n\to\infty} (-1)^n a_n = 0$

No ↓

Integral test
Known antiderivative ?

→ Convergent Condition
$\int_1^\infty f(x)dx < \infty$

No ↓

Long term behavior

Comparison test
$\Sigma a_k < \Sigma b_k < L$
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Limit Comparison test
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$\Sigma b_k < \infty$: convergent Σa_k
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Ratio/root test
$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$
$\lim_{n\to\infty} (a_n)^{1/n} = \rho < 1$
convergent

+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

Evaluate

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Integral test

$$\int_2^\infty \frac{1}{x \ln x} dx = \lim_{N\to\infty} \int_2^N \frac{d(\ln x)}{\ln x}$$
$$= \lim_{N\to\infty} [\ln|\ln x|]_2^N$$
$$= \infty$$

Alternating series

Definition: Alternating series

An infinite series of the form of

(a) $\sum_{k=1}^{\infty}(-1)^k a_k$ (b) $\sum_{k=1}^{\infty}(-1)^{k+1} a_k$

where $a_k > 0$

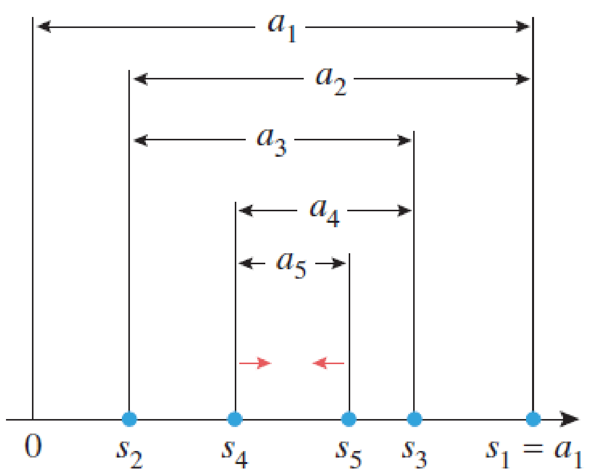
Theorem (*Alternating Series Test*): Alternating series with diminishing oscillation converges

An alternating series converges if the following two conditions are satisfied:

(a) $a_K > a_{K+1} > a_{K+2} > a_{K+3} > \cdots$

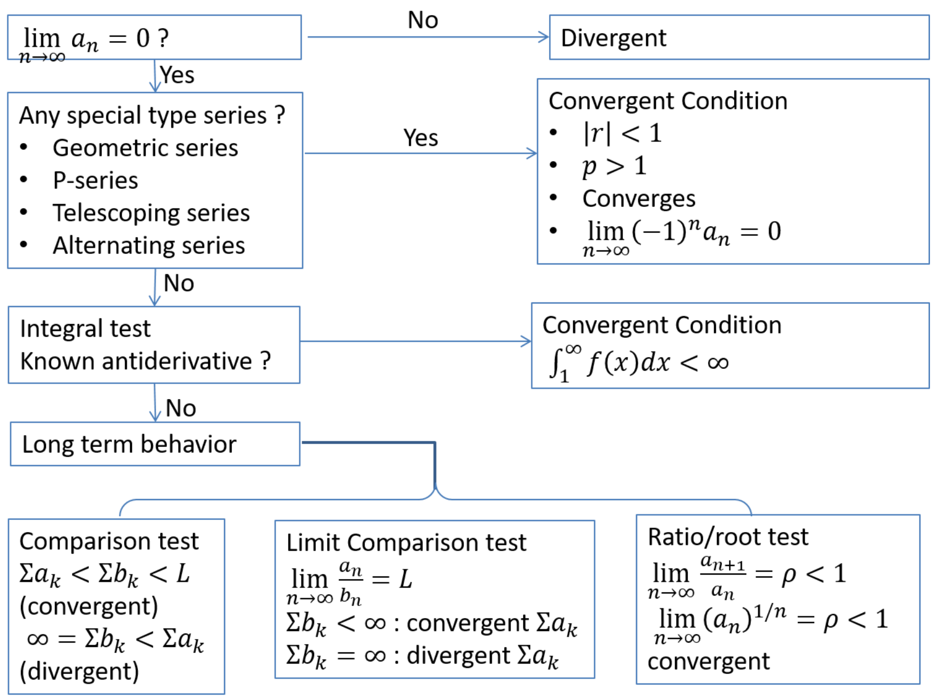
(b) $\lim_{k\to\infty} a_k = 0$

- $s_1, s_3, s_5, \cdots, s_{2n-1}, \cdots$: $\{s_{2n-1}\}$ is decreasing sequence bounded below by 0.
- $s_2, s_4, s_6, \cdots, s_{2n}, \cdots$: $\{s_{2n}\}$ is increasing sequence bounded above bya_1.
- Since bounded monotone sequences converge both $\{s_{2n-1}\}$ and $\{s_{2n}\}$ converge.
- $\lim_{n\to\infty}(s_{2n} - s_{2n-1}) = \lim_{n\to\infty} a_{2n} = 0$
- $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n-1} \Rightarrow \lim_{n\to\infty} s_n$ converges



Week 10

Alternating series



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$$

- $\lim_{n\to\infty} \frac{1}{n} = 0$ (convergent)
- $\lim_{n\to\infty} \frac{n}{n+1} = 1$ (divergent)
- $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$ (convergent)

Absolute convergence

Definition: Absolute convergence for general mixed sign series

- A series $\sum u_k$(u_k be positive or negative) is said to ***converge absolutely*** if $\sum |u_k|$ converges
- A series $\sum u_k$(u_k be positive or negative) is said to ***converge conditionally*** if $\sum u_k$ converges but $\sum |u_k|$ diverges

Absolute convergence Theorem

If $\sum |u_k|$ converges then $\sum u_k$ converges

- If $\sum u_k$ diverges, then $\sum |u_k|$ diverges
- If $\sum |u_k|$ converges then $\lim_{k\to\infty} u_k = 0$

Write $\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty}[(u_k + |u_k|) - |u_k|]$

- Since $0 \le u_k + |u_k| \le 2|u_k|$ and $\sum_{k=1}^{\infty} 2|u_k| = 2\sum_{k=1}^{\infty}|u_k|$ (converges), by comparison test $\sum_{k=1}^{\infty}(u_k + |u_k|)$ converges
- Now that $\sum_{k=1}^{\infty}(u_k + |u_k|)$ converge, and $\sum_{k=1}^{\infty}|u_k|$ converges, by the limit theorem of series, $\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty}[(u_k + |u_k|) - |u_k|]$ converges
  - $\sum a_k = S_a$ and $\sum b_k = S_b \Rightarrow \sum (a_k + b_k) = S_a + S_b$

Exercise

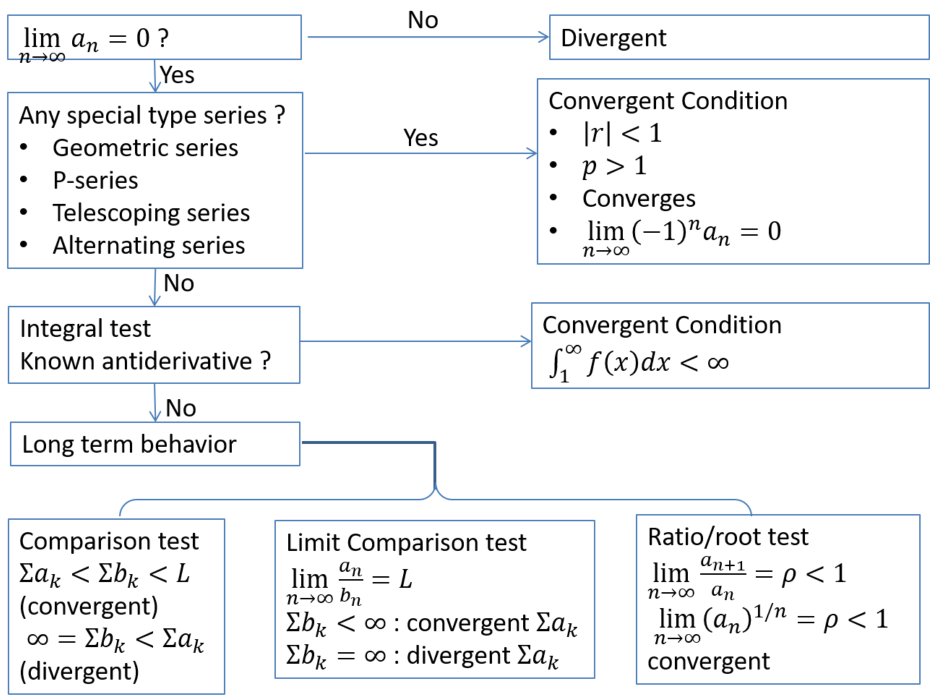
Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converge?
If so, how?

$\sum_{n=k}^{\infty} \frac{\ln n}{n^3} \leq \sum_{n=k}^{\infty} \frac{n}{n^3}$ for some k

$\sum_{n=1}^{k-1} \frac{\ln n}{n^3} + \sum_{n=k}^{\infty} \frac{\ln n}{n^3} \leq \sum_{n=1}^{k-1} \frac{\ln n}{n^3} + \sum_{n=k}^{\infty} \frac{1}{n^2} < \infty$

Week 10

Exercise



+/- terms but Not alternating
$\Rightarrow$ Absolute convergence

Does $\sum_{n=1}^{\infty} \frac{2+\sin n}{n^2}$ converge?
If so, how?

Does $\sum_{n=1}^{\infty} \frac{2+\sin n}{n}$ converge?
If so, how?

Does $\sum_{n=2}^{\infty} \frac{2+\sin n}{n \ln n}$ converge?
If so, how?

Since

$\sum_{n=1}^{\infty} \frac{1}{n^2} \le \sum_{n=1}^{\infty} \frac{2+\sin n}{n^2} \le \sum_{n=1}^{\infty} \frac{3}{n^2}$

and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{3}{n^2} < \infty$,

$\sum_{n=1}^{\infty} \frac{2+\sin n}{n^2} < \infty$

Since $\sum_{n=2}^{\infty} \frac{k}{n} = \infty$, $\sum_{n=1}^{\infty} \frac{2+\sin n}{n} = \infty$

Since $\sum_{n=2}^{\infty} \frac{k}{n \ln n} = \infty$, $\sum_{n=1}^{\infty} \frac{2+\sin n}{n^2} = \infty$

Week 10

Exercise

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. Circle the true statement(s):

If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

If $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} b_n$ is divergent.

If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

There is not enough information.

Exercise

Does $\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)}$ converge?

If so, how?

Denote $f(x) = O\big(g(x)\big)$ *iff* $\lim_{n\to\infty} \frac{f}{g} = L < \infty$

$\frac{n}{(n+1)(n+2)} \sim O\left(\frac{1}{n}\right)$ i.e. $\lim_{n\to\infty} \frac{\frac{n}{(n+1)(n+2)}}{\frac{1}{n}} = 1$

Since $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, $\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)} = \infty$

Exercise

Which of the following statements is true for the following series?

(I) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ (II) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^3}$ (III) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{(-1)^n n}$

(a) I and III converge conditionally, and II diverges.

(b) I converges conditionally, II converges absolutely, and III diverges. $\leftarrow$ correct

(c) I and II converge conditionally, and III diverges.

(d) I, II, and III converge conditionally.

(e) I, II, and III converge absolutely.

Week 10

Exercise

Does $\sum_{n=1}^{\infty} \frac{\sin n}{n^2+n+1}$ converge?
If so, how?

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2+n+1}\right| \le \sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$$

$$\le \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$< \infty$$

Absolutely convergent -> convergent

Week 10

Exercise

Does $\sum_{n=1}^{\infty} \frac{2^n}{3^n+n^2}$ converge?
If so, how?

Denote $f(x) = O\big(g(x)\big)$ *iff* $\lim_{n\to\infty} \frac{f}{g} = L < \infty$

$\frac{2^n}{3^n+n^2} \sim O\left(\frac{2^n}{3^n}\right)$ i.e. $\lim_{n\to\infty} \frac{\frac{2^n}{3^n+n^2}}{\frac{2^n}{3^n}} = 1$

Since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n < \infty, \sum_{n=1}^{\infty} \frac{2^n}{3^n+n^2} < \infty$

Exercise

(10 points) Determine whether the following series is absolutely convergent, conditionally convergent, or divergent. Show all work, as illustrated in class, by naming the test(s), applying the test(s), and drawing the correct conclusion(s).

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$$

$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1} < \infty$ (Alternating series)

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n n}{n^2+1}\right| = \sum_{n=1}^{\infty} \frac{n}{n^2+1} = \infty$ ($\frac{n}{n^2+1} \sim O\left(\frac{1}{n}\right)$ (limit comparison) and $\sum \frac{1}{n} = \infty$)

Therefore,

$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ converges conditionally

Exercise

Determine whether the series converges or diverges. Justify.

$$\sum_{n=1}^{\infty} n^3 \sin\left(\frac{1}{n^3}\right)$$

Determine whether the series converges or diverges. Justify.

$$\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n^3}\right)$$

$$n^3 \sin\left(\frac{1}{n^3}\right) = \frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n^3}} \to 1$$

By divergence test, $\sum_{n=1}^{\infty} n^3 \sin\left(\frac{1}{n^3}\right) = \infty$

$$n \sin\left(\frac{1}{n^3}\right) = \frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n}} \sim O\left(\frac{\frac{1}{n^3}}{\frac{1}{n}}\right) = O\left(\frac{1}{n^2}\right) \text{ i.e. } \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^3}\right)}{\frac{1}{n}} = 0 < \infty$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n^3}\right) < \infty$